CONCISE SPHERICAL TRIGONOMETRY WITH APPLICATIONS

Hammond

Concise Spherical

HOUGHTON MIFFLIN COMPANY

Trigonometry

WITH APPLICATIONS AND REVIEWS OF SOLID GEOMETRY AND PLANE TRIGONOMETRY

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PREFACE

The necessarily increased air-mindedness of the present generation is making a technical acquaintance with the subject of spherical trigonometry of practical importance. To know the nature and methods of solution of "course and distance" problems is as significant a part of the mental equipment of the student today as to know how to find heights of distant objects and distances of inaccessible places in plane trigonometry.

The cultural applications of spherical trigonometry have always been important. Anyone professing curiosity concerning the phenomena of the universe about him should list as of fundamental importance a working knowledge of the motions of the earth as a basis for understanding systems of time measurement and methods of fixing positions on the earth's surface.

The teaching of spherical trigonometry has suffered from two extreme points of view. On the one hand, because there are many different ways of solving general spherical triangles, it has been assumed that all methods must be developed and no one method emphasized. This has necessitated the teaching of numerous complicated formulas whose use is confined solely to spherical trigonometry. On the other hand (and partly because of the above), the need for the rapid solution of certain typical spherical triangles in navigation has made necessary the employment of some entirely formal method. Several such methods,* based almost entirely on the proper manipulation of specially prepared tables, will quickly solve a navigator's problems without conscious reference to the geometrical concepts of spherical trigonometry. This is, of course, as it should be; decisions at sea and in the air must be based on the facts of the ship's position, and the facts must be found at once. But to

* The following publications contain the most frequently used tabular methods of navigation. The abbreviation "H.O." indicates a publication of the Hydrographic Office, United States Navy Department.

Tables of Computed Altitude and Azimuth. H.O. 214. Devised by Commander R. H. Knight, U.S.N. (Ret.) and Commander R. E. Jasperson, U.S.N. Washington, D.C.: Government Printing Office, 1939–1941.

Dead Reckoning Altitude and Azimuth Table. H.O. 211. Devised by Commander A. A. Ageton, U.S.N. Washington, D.C.: Government Printing Office, 1940.

Navigation Tables for Mariners and Aviators. H.O. 208. Devised by Commander J. Y. Dreisonstok, U.S.N. (Ret.). Washington, D.C.: Government Printing Office, 1942.

Cosine-Haversine Formula of Marcq Saint-Hilaire. H.O. 171. Washington, D.C.: Government Printing Office, 1915.

Star Altitude Curves. Devised by Lieutenant Commander P. V. H. Weems, U.S.N. (Ret.). Annapolis, Maryland: Weems System of Navigation, 1938.

advocate teaching at the outset one of the several excellent formal methods of solution of navigation triangles without any previous instruction in the concepts and principles of spherical trigonometry would be analogous to teaching integration directly and solely by means of integral tables.

Granting then that the study of any of the formal methods of solving certain spherical triangles should be preceded by a course in spherical trigonometry, the necessity of developing all possible methods of the mathematical solution of spherical triangles does not follow. One method, the fundamental right spherical triangle method, will solve all spherical triangles with certainty, completeness, and simplicity. By this method all the fundamental concepts of spherical trigonometry can be amply demonstrated and emphasized. *Accordingly, in the text proper the solution of all spherical triangles is accomplished by the one fundamental right triangle method.* The student's gain in not having to derive many intricate formulas, which are of no value outside of spherical trigonometry, is immensely greater than any loss from a possible slight increase in computation. Solving general spherical triangles by dropping a perpendicular clarifies the student's geometrical conception of the various possible aspects of spherical triangles. The student can reasonably be held responsible for the derivation of all formulas necessary for the solution of any triangle (an unreasonable point of view in the case of the several special methods). For the student who might be particularly interested in special methods of general triangle solution, Appendix II is provided. Here the usual formulas necessary for these special methods are derived with as much motivation as possible. The relative merits of all solutions are briefly discussed.

By having presented to him essentially but one method of solving all spherical triangles (namely, the Napier's Rules solution of right spherical triangles), the student can be expected to base this one method firmly upon his solid geometry. This the present text facilitates by listing in the Introduction all the solid geometry definitions and theorems which are in any way involved in the following treatment of spherical trigonometry. Each concept is represented by a sketch, and the proofs of theorems are briefly outlined. References to specific parts of the Introduction are made at the appropriate points in the text.* Spherical trigonometry is much more closely related to solid geometry than to plane trigonometry, from which it need borrow only

* Sections 7 and 8 of Part *A* of the Introduction contain material which does not appear in the usual solid geometry syllabus.

the definitions of the six trigonometric functions and their five interrelations which are rational. For this reason spherical trigonometry can logically be presented apart from plane trigonometry, provided it is adequately based upon solid geometry. All the material in plane trigonometry needed in the text proper and in the appendixes is given in the Introduction following the solid geometry references.

The chapters on terrestrial applications and celestial applications are given considerable space. Not only do these chapters present the important applications of spherical triangle solution, but the concepts therein discussed are of cultural importance to any layman. The experience of the author has abundantly indicated that the college or high school student has very confused and inexact ideas about such simple concepts as the small-circle distance between two points on the earth's surface in the same latitude, and the distinction between sidereal and solar time. The grammar-school course in geography is too far in the student's past and was necessarily too elementary to provide him with a permanently satisfactory grasp of the geometrical aspects of the earth's surface and the earth's motions in the heavens. These chapters on terrestrial and celestial applications can well be assigned for supplementary reading while the technique of triangle solution is being developed.

In brief, the following are the aims of the present text on spherical trigonometry:

1. To base the subject firmly upon solid geometry by means of references to solid geometry in the Introduction;
2. To present *one method* of solving all spherical triangles, *i.e.*, the right triangle method. This method can then be easily remembered and thoroughly mastered. It will never lead to uncertainties;
3. To demand of the student an arbitrary system of computation in the interest of clarity and accuracy;
4. To provide completely worked-out examples as guides for the solution of assigned problems;†
5. To place all material not essential for the first reading of the subject in the appendixes and to include in the appendixes all material about which the student might reasonably be curious. Special methods of general triangle solution, a complete discussion of ambiguous cases, instruments used in observing the data of spherical trigonometry problems, and almanacs are considered in this category;
6. To provide, in the chapters on terrestrial and celestial applications,

† Bowditch Tables [*Useful Tables*: U.S. Hydrographic Office] have been used in this text.

a reasonable discussion of the culturally important subjects of the earth's surface-geometry and the motions of the earth in the heavens.

The illustrations and diagrams of some of the navigation instruments described in Appendix III were drawn from photographs supplied by the following firms:

- Keuffel and Esser Company of Hoboken, New Jersey — sextant and transit with solar attachment;
- Bausch and Lomb Optical Company of Rochester, New York — bubble octant;
- T. S. and J. D. Negus of New York, New York — azimuth circle;
- Kelvin-White Company of Boston, Massachusetts — magnetic compass.

This courteous helpfulness has materially simplified the writing of this appendix and is greatly appreciated.

A very considerable part of the satisfaction in this enterprise is identified with the stimulation and generous help of many friends. Captain S. P. Fullinwider, U.S.N., Head of the Department of Mathematics at The Naval Academy, has been an encouraging and informative adviser on sources of technical information. Captain J. F. Hellweg, U.S.N., Retired, Superintendent of The Naval Observatory, has sanctioned the inclusion of the appendix on the *Nautical Almanac* and the *Air Almanac* and has helpfully advised on the Star Chart. Lieutenant Commander Paul Miller, U.S.N., Retired, has expertly advised on innumerable matters pertaining to navigation. The same has been true of Associate Professor William A. Conrad in the details of astronomy. To these two specialists the author is particularly indebted for help in properly dealing with the applications of spherical trigonometry. Among a host of other helpful colleagues Mr. Walton H. Sears, Jr. is to be singled out for his many resourceful suggestions and for his painstaking interest in reading proof and in improving the accuracy of the illustrative examples and problems.

JACQUES HAMMOND

ANNAPOLIS, MARYLAND

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PART FOUR. The Transit and Its Solar Attachment

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Introduction

1. Explanation

The following is a development of the concepts and theorems of solid geometry and plane trigonometry which are necessary for a complete understanding of spherical trigonometry. This introductory review is important in so far as it aids in understanding the chapters which follow. References are made to specific parts of this Introduction at appropriate points in the text. Consequently, the student (unless notably unfamiliar with the geometry of three dimensions, particularly as applied to the sphere, and the fundamentals of plane trigonometry) can well skip this Introduction for the present and refer to it when and as it becomes necessary.

A. DEFINITIONS AND THEOREMS FROM SOLID GEOMETRY

2. Planes; Parallel Planes

a. Determination: A plane is determined by three points, by two intersecting lines, by a line and a point not on the line, or by two parallel lines. (See Figure 1.)

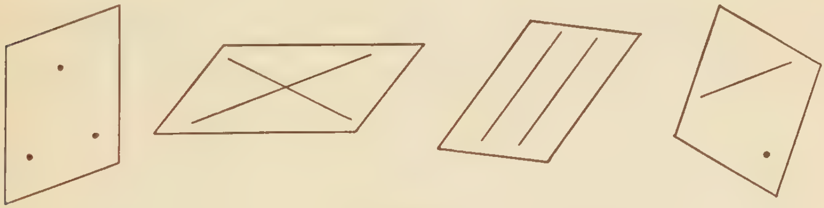

Figure 1

b. Representation: A plane is infinite in extent in all directions, but is conveniently pictured as a parallelogram.

c. Definition: Two planes are ***parallel planes*** if they do not intersect no matter how far extended. (See Figure 2.)

d. Construction: Through a given point not in a given plane to construct the unique plane parallel to the given plane. (See Figure 3.)

(1) x and y are any two lines in P, the given plane.

FIGURE 2

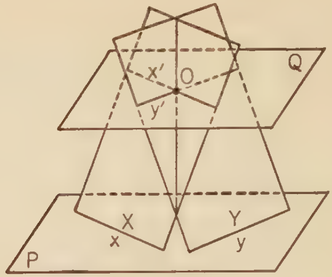

FIGURE 3

(2) X and Y are the planes determined by the given point O and the lines x and y, respectively.

(3) x' and y' are lines through O in X and Y, respectively, parallel to x and y, respectively.

(4) The required plane Q is the plane of x' and y'. Suppose Q should meet P in a line z. Then z would cut x or y, say x. Let Z be the supposed intersection of line z with line x. Then point Z must be in the plane X and in the plane Q. Hence Z would be on the line x', the line of intersection of the planes X and Q. Therefore, x' would intersect x in the point Z. But $x' \parallel x$.

3. Line and Plane Perpendicular

a. DEFINITION: A line and plane shall be said to be mutually perpendicular when the line is perpendicular to all lines which are in the plane and which pass through the point of intersection of the line and the plane.

In Figure 4 if the line $k \perp$ plane P at point O, then all the angles at O between k and lines in P are $= 90°$, and conversely.

b. THEOREM: A line and a plane are mutually perpendicular when the

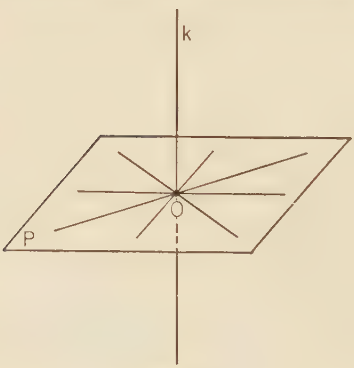

FIGURE 4

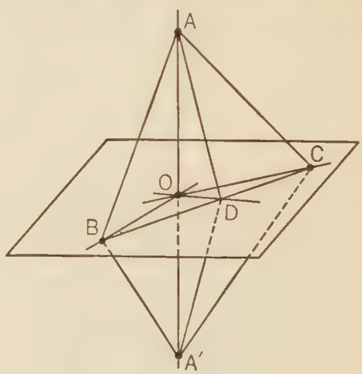

FIGURE 5

line is perpendicular to any two lines which are in the plane and which pass through the point of intersection of the line and the plane. (See Figure 5.)

(1) In the figure $\measuredangle\ AOB = \measuredangle\ AOC = 90°$. To prove that any other angle, such as AOD, where OD is also in plane P, is 90°.

(2) Extend AO its own length to A'. OD is any line in P through O not identical to the two given lines OB, OC. BDC is any line in P cutting OB, OD, and OC in B, D, and C, respectively.

(3) $\triangle\ AOB \cong \triangle\ A'OB$ and $\triangle\ AOC \cong \triangle\ A'OC$ by s.a.s. (side, angle, side) by hypothesis and construction.

(4) $\triangle\ ABC \cong \triangle\ A'BC$ by s.s.s. from (3).

(5) $\triangle\ ABD \cong \triangle\ A'BD$ by s.a.s. from (3) and (4).

(6) $\triangle\ AOD \cong \triangle\ A'OD$ by s.s.s. from (2) and (5).

(7) $\measuredangle\ AOD = \measuredangle\ A'OD = 90°$.

c. THEOREM: A line perpendicular to one of two parallel planes is perpendicular to the other plane. (See Figure 6.)

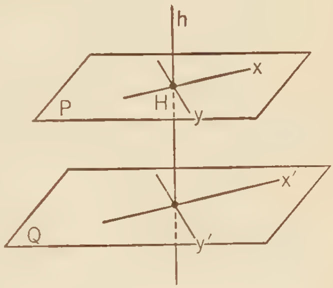

FIGURE 6

In the figure, plane $P \parallel$ plane Q and line $h \perp P$ at point H. Let x, y be any two lines in P through H. Let x', y' be the lines in which plane Q intersects the planes determined by h and x and h and y, respectively. Then $x' \parallel x$ and $y' \parallel y$, as the lines of each pair are in a plane and never meet, since they lie in parallel planes. $\therefore x'$ and y' are each $\perp h$, as a line in the plane of two parallel lines and $\perp$ one of them is $\perp$ the other. $\therefore h \perp Q$, by 3 *b*.

d. CONSTRUCTION: Through a given point to construct the unique plane perpendicular to a given line (1) when the given point is on the given line, (2) when the given point is not on the given line.

(1) In Figure 7 *a*, pass any two planes through the given line k. In each

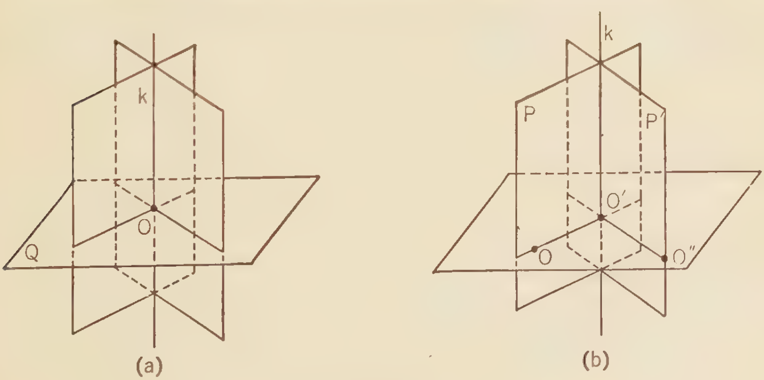

FIGURE 7

of these two planes construct the perpendicular to k at the given point O on k. The plane of these two perpendiculars is the required plane.

(2) In Figure 7 b, construct the plane P determined by the given line and point. In this plane drop a perpendicular from the given point O to the given line k, intersecting k at O'. Pass any other plane P' through k. In P' construct the perpendicular $O'O''$ to k at O'. The plane of the lines OO', $O'O''$ is the required plane.

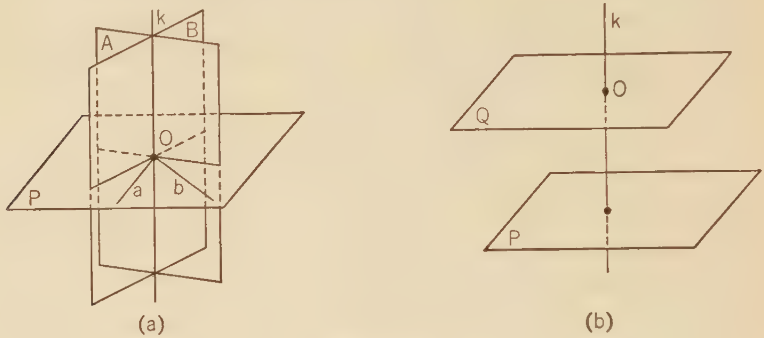

FIGURE 8

e. CONSTRUCTION: Through a given point to construct the unique line perpendicular to a given plane (1) when the given point is in the given plane, (2) when the given point is not in the given plane.

(1) In Figure 8 a, a and b are any two lines drawn in the given plane P, and through the given point O. A and B are the unique planes constructed perpendicular to lines a and b, respectively, at O. Line k is the intersection of planes A and B and is the required line, since, by construction, it makes 90° angles with lines a and b in plane P at O.

(2) In Figure 8 b, plane Q is constructed, by 2 d, parallel to the given plane P, and through the given point O. Then line k is constructed perpendicular to Q at O, by 3 e (1). Then k is perpendicular to P, by 3 c.

f. THEOREM: The locus in space of points equidistant from the extremities of a line segment is the plane perpendicularly bisecting the line segment.

(1) In Figure 9, P is given on the perpendicularly bisecting plane. Then $XP = YP$, by triangles congruent by s.a.s.

(2) P' is given equidistant from X and Y. Then the angles at M are equal and equal 90°, by triangles congruent by s.s.s. Hence, P' is in the perpendicularly bisecting plane.

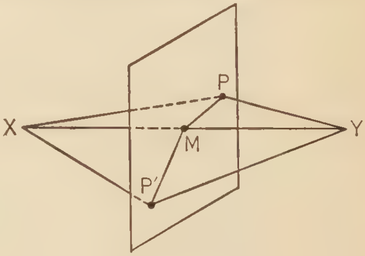

FIGURE 9

4. Intersecting Planes

a. DEFINITION: A ***dihedral angle*** is any one of the four openings between two intersecting planes. The planes are called the ***faces*** of the dihedral angle. The line of intersection of the faces is called the ***edge*** of the dihedral angle. (See Figure 10.)

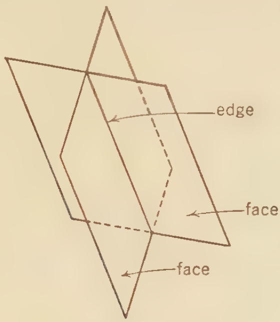

FIGURE 10

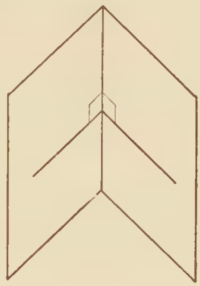

FIGURE 11

b. DEFINITION: The ***plane angle of a dihedral angle*** is the plane angle formed at any point on the edge of the dihedral angle by two intersecting lines, one in each face of the dihedral angle, and each perpendicular to the edge at this point. The plane angle obviously measures the opening between the two planes, and, hence, the ***measure of the dihedral angle*** is defined as the measure of its plane angle. (See Figure 11.)

c. THEOREM: All planes containing a line perpendicular to a given plane are themselves perpendicular to the given plane.

In Figure 12, line k is given perpendicular to the plane P. R is any plane containing k and meeting plane P in the line a. At the point O, the intersection of the line k and plane P, the line b is drawn in P and perpendicular to line a. Then line k is perpendicular to lines a and b. The angle between

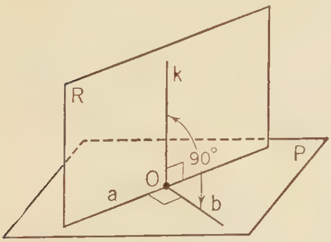

FIGURE 12

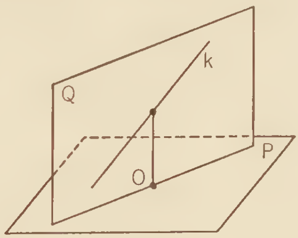

FIGURE 13

k and b is the plane angle of the dihedral angle between P and R and is equal to 90°.

d. THEOREM. Through a given line not perpendicular to a given plane to construct the unique plane perpendicular to the given plane. (See Figure 13.)

From any point on the given line k construct the unique perpendicular to the given plane P. The required plane Q is the plane determined by this constructed perpendicular and the given line k, by 4 *c*.

e. THEOREM. If two planes are perpendicular, then any line in one of them perpendicular to the line of intersection of the two planes is perpendicular to the other plane. (See Figure 14.)

Given that planes P and Q are perpendicular and that the line k in Q is perpendicular to the line a, the intersection of P and Q. At point O, the intersection of k and P, and in P draw the line b perpendicular to a. Then the angle of k and b equals 90°, as it is the plane angle of the dihedral angle of P and Q, which is given equal to 90°.

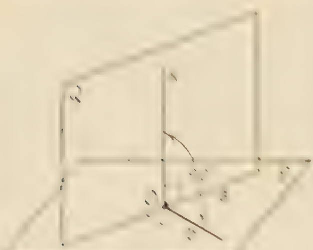

FIGURE 14

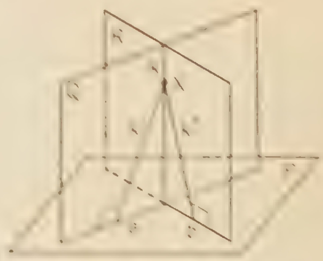

FIGURE 15

f. THEOREM. The line of intersection of two planes, each perpendicular to a third plane, is itself perpendicular to this third plane. (See Figure 15.)

Planes Q and R are given ⊥ plane P and meet P in lines a and b, respectively. Line k is the intersection of planes Q and R. Let X be any point on k (but not in P). Through X draw line k' and line k'' ⊥ respectively to a and b. Then both k' and k'' are ⊥ P by 4 *e*, as k' is in Q and k'' in R. Since there is but one perpendicular to a plane from a point (X), $k' \equiv k''$, and therefore k' lies also in R as well as in Q. Hence, $k' \equiv k'' \equiv k$, which is therefore ⊥ P.

5. Polyhedral Angles

a. DEFINITION. When three or more planes intersect so that (1) they have one and only one point in common, and (2) they intersect some other plane in a convex polygon, then the opening at the common point of the planes in the direction of the plane of the convex polygon is called a ***convex polyhedral angle.*** When the number of intersecting planes is three, a convex trihedral angle or a ***trihedral angle*** is formed. The common point is called the ***vertex*** of the polyhedral angle; the plane angles at the vertex are called the ***face angles*** of the polyhedral angle; the angles between the planes, the ***dihedral angles*** of the polyhedral angle; and the intersections of the planes or ***faces***, ***edges.***

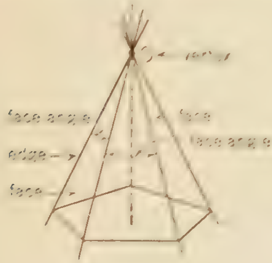

FIGURE 16

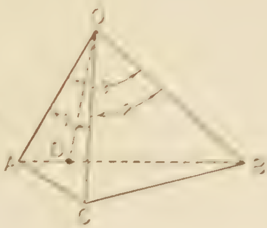

FIGURE 17

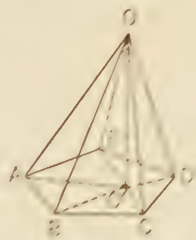

FIGURE 18

b. THEOREM: The sum of two face angles of a trihedral angle is greater than the third face angle.

In Figure 17, angle 3 > angle 2. In the face of angle 3 angle DOB is constructed equal to angle 2 and OD is made equal to OC. A, B are intersections with the edges OA, OB, respectively, of some plane through C and D. Then $BD = BC$, as triangles BOD, BOC are congruent, but $AB < AC + CB$ and, therefore, $AD < AC$. Hence angle AOD < angle 1 by plane geometry. Therefore, angle 3 < angle 2 + angle 1.

c. THEOREM: The sum of the face angles of any convex polyhedral angle is less than two straight angles.

In Figure 18, $ABCDE$ is the convex polygon formed by the faces of the polyhedral angle cutting some other plane. O' is an interior point of the polygon. Now, the sum of the angles of the triangles with vertices at O equals the sum of angles of the triangles with vertices at O'. But, by the above theorem, $\angle ABO + \angle CBO > \angle ABC$, etc. Hence, the sum of the angles not at O of triangles with vertices at O > the sum of the angles not at O' of triangles with vertices at O'. Therefore, the sum of angles at O of triangles with vertices at O < the sum of angles at O' of triangles with vertices at $O' = 360°$.

6. Great Circles on a Sphere, Pole, and Polar

a. DEFINITIONS: The circle of intersection of a plane and a sphere is called a ***great circle*** (Figure 19 *a*) if and only if the plane of intersection contains the center of the sphere, and a ***small circle*** (Figure 19 *b*) when this is not the case.

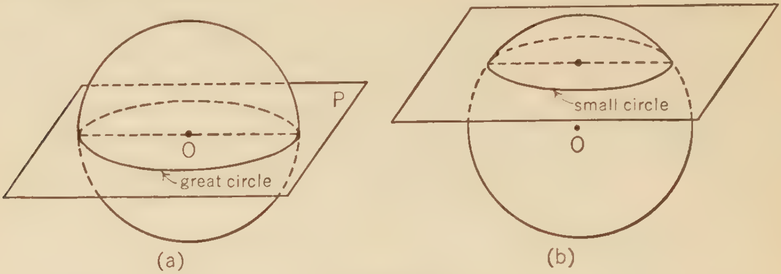

FIGURE 19

b. DEFINITIONS: When two great circles of a sphere intersect, any one of the four openings on the sphere at the point of intersection and between the two great circles is called a ***spherical angle.*** The ***measure of the spherical angle*** is the measure of the dihedral angle of the planes of the two great circles. Those portions of the surface of the sphere lying between halves of each great circle are called ***lunes.***

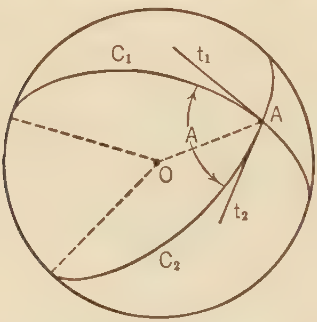

FIGURE 20

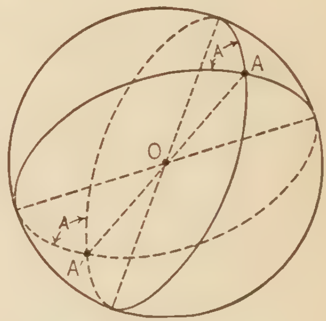

FIGURE 21

In Figure 20, C_1 and C_2 are two great circles intersecting at A. The spherical angle A is marked with the curved arrow. OA (O is the center of the sphere) is the edge of the dihedral angle of the planes of the two circles C_1, C_2, and t_1, t_2 are the tangents at A to C_1, C_2, respectively. *Then spherical angle* A = *dihedral angle* $\mathrm{C_1 - OA - C_2}$ = *plane angle* $\mathrm{t_1At_2}$.

c. THEOREM: Two great circles intersect one another at the same respective angles at two diametrically opposite points.

Since by definition great circles are formed by planes through the center of the sphere, the planes of the two great circles must intersect in a diameter at opposite ends of which the great circles intersect one another. The angles at one point of intersection are equal to the respective angles at the other point, because each member of a pair of respective angles is measured by the dihedral angle of that pair. (See Figure 21.)

d. DEFINITIONS: A ***pole*** of a great-circular arc is either one of the two points at which the sphere is pierced by the diameter perpendicular to the plane of the great-circular arc.

In Figure 22, P and P' are each poles of the great circle C.

The ***polar*** great circle of a given point on a sphere is the great circle of which the given point is a pole, or it is the great circle in which the sphere is cut by the diametral plane perpendicular to the diameter through the given point.

In Figure 22, C is the polar great circle of each of the points P and P'.

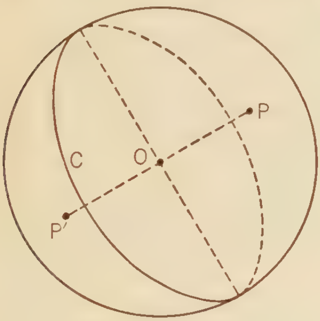

FIGURE 22

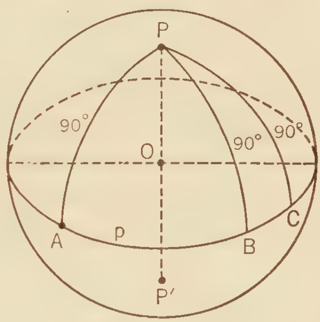

FIGURE 23

e. THEOREM: The polar of a point on a sphere is the locus on the sphere of all points 90° of arc (measured on great circles) from the given point.

(1) In Figure 23, p is the polar of P and its diametrically opposite point P'. Hence, p is in the plane perpendicularly bisecting PP'. Hence, all central angles between either P or P' and any point on p are right angles, as are the corresponding great-circular arcs subtended by these central angles.

(2) If a point A on the surface of the sphere is 90° of arc from either P or P', it is also 90° of arc from the other of these two diametrically opposite points P and P'. Hence, A is equidistant from P and P' and therefore, by 3 *f*, in the plane perpendicularly bisecting PP', which plane intersects the sphere in the polar, p, of P.

f. THEOREM: All great circles through a given point cut the polar of the given point perpendicularly.

In Figure 24, PO is $\perp$ plane of polar of P, by definition of pole. Hence, also planes of great circles through P are $\perp$ this plane, by 4 *c*.

g. THEOREM: The intersections of the polars of two points on a sphere are poles of the great-circular arc between the two points.

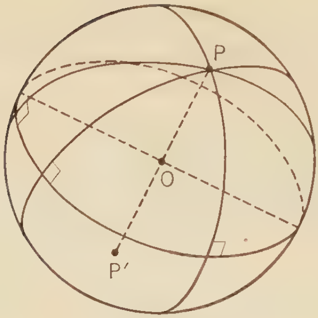

FIGURE 24

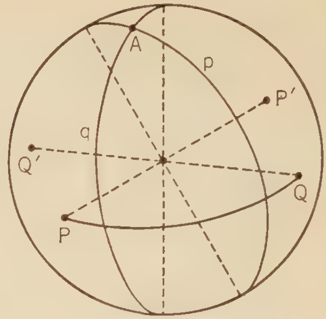

FIGURE 25

In Figure 25, A is one point of intersection of p and q, the polar arcs of points P and Q, respectively. Hence, A is 90° of arc from both P and Q, or P and Q are each 90° of arc from A. Therefore, by 6 *e*, P and Q are each on the polar arc of A. But P and Q in general determine but one great circle, which is therefore the great circle polar to A.

h. THEOREM: A point 90° of great-circular arc from each of two points (not at opposite ends of a diameter of the sphere) of a given great circle is a pole of the given great circle.

In Figure 26, arcs PA, PB are given 90°. Hence, angles POA, POB are right angles. Hence, by 3 *b*, $PO \perp$ the plane of the given great circle of A and B. Hence, P is a pole of this given great circle, by definition of pole and polar.

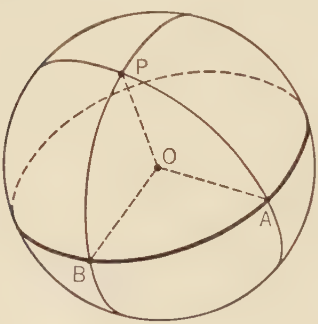

FIGURE 26

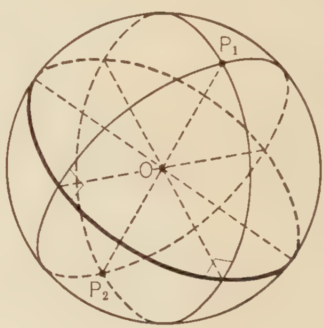

FIGURE 27

i. THEOREM: Two great circles, each perpendicular to a third great circle, intersect one another in the poles of this third great circle, and hence all great circles perpendicular to a given great circle pass through its pole. (See Figure 27.)

The planes of the two great circles perpendicular to the third great circle are each perpendicular to the plane of this third great circle, by 6 *b*. Hence, by 4 *f*, the diameter of the sphere common to these two planes is also perpendicular to the plane of the third great circle, and will therefore intersect the sphere in the poles of this third great circle, by 6 *d*.

j. THEOREM: A spherical angle has the same measure as that portion of the arc of the polar of its vertex which is included between the sides of the angle.

In Figure 28, XY is that portion of the arc of the polar of A which is subtended by the sides of the spherical angle at A. But XY has the same measure as the angle at the center of the sphere formed by the radii to X and Y. These radii are perpendicular to the radius to A, because A is the pole of XY. Hence, this central angle subtended by XY is equal to the plane angle of the dihedral angle of the spherical angle A.

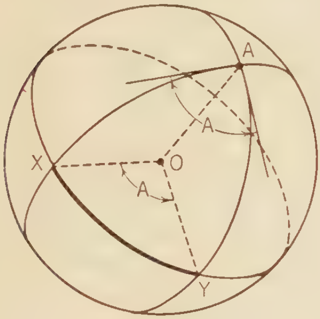

FIGURE 28

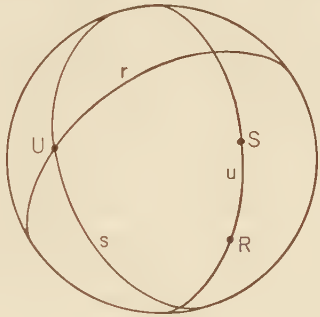

FIGURE 29

k. THEOREM: The poles of a great circle lie on the polar of any point on it, or, if one great circle contains the pole of a second, then the second great circle contains the pole of the first.

In Figure 29, u is a given great circle, R is any point on u, and r is the polar or R. Let S be any other point on u. Let the polar s of S intersect r in U. Then, by 6 *g*, the polar of U is u. Hence, the pole of u is U, which, therefore, lies on the polar of R.

7. Small Circles on a Sphere *

a. DEFINITION: A ***small circle*** on a sphere is the intersection with the sphere of a plane not through the center of the sphere. (See Figure 19 *b*.)

* As previously mentioned in the Preface, sections 7 and 8 contain material which does not appear in the usual solid-geometry syllabus. They are included here because of their indispensability in explaining the solar attachment to the transit (cf. Appendix III, section 36). This instrument, though not often used, is interesting because it demonstrates automatically some of the fundamental applications of the celestial sphere. The material of these two sections, moreover, can justly be considered significant *per se*. The interested student will quickly perceive the plane-geometry analogues that exist for many of the concepts and theorems discussed in these sections.

b. Definition: The ***pole of a small circle*** of a sphere is the nearer extremity of the sphere's diameter which is perpendicular to the plane of the small circle.

c. Theorem: If two small circles on a sphere intersect in two points, these two points symmetrically straddle the great circle through the poles of the small circles.

(1) In Figure 30, c_1, c_2 are the small circles meeting in X and Y, and P_1, P_2 are the respective poles of c_1, c_2. C is the great circle through P_1 and P_2.

(2) Plane of C is $\perp$ planes of c_1 and c_2, by 4 *c* and 7 *b*.

(3) Plane of C is $\perp$ XY, by 4 *f*.

(4) Plane of C perpendicularly bisects XY, by 3 *f* and 3 *d*, since O is equidistant from X and Y.

(5) Therefore, C is symmetrically straddled by X and Y, by 3 *f* and by the fact that equal chords of a sphere subtend equal great-circular arcs of the sphere.

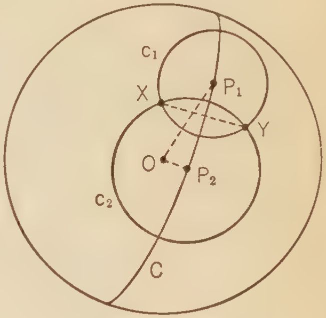

Figure 30

d. Definition: A point on a sphere will be said to be ***outside a given small circle*** on the sphere, if it is on the larger of the two unequal parts of the sphere defined by the small circle.

8. Tangent Lines and Tangent Circles on a Sphere *

a. Definition: A straight line tangent to a circle, great or small, on a sphere will be said to be ***tangent to the sphere*** at the point of tangency with the circle. (See Figure 31.)

b. Theorem: A tangent to a sphere touches the sphere in one and only one point.

(1) The tangent to the sphere cannot touch the sphere again on the circle to which it is tangent, by definition of a line tangent to a circle.

(2) The tangent cannot touch the sphere at any point on the sphere not on the circle of tangency, as the tangent lies in the plane of this circle which is the locus of points common to the plane and the sphere.

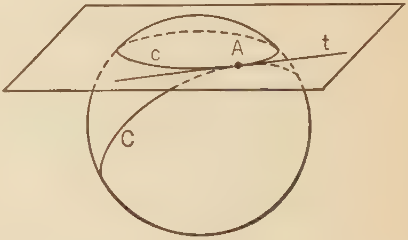

Figure 31

c. Definition: Two circles on a sphere and passing through a point A on the sphere will be said to be ***circles tangent to one another at the***

* See note on page 11.

point *A* if the two circles possess a common tangent, a tangent to the sphere, at the point *A*. (See Figure 31.)

d. THEOREM: The line of intersection of the planes of two tangent circles on a sphere is the line of common tangency.

Since the common tangent lies in the plane of each circle, it must be the line of intersection of the planes of the two circles.

e. THEOREM: Tangent circles on a sphere have no point other than the point of tangency in common.

Points common to the two circles must lie in the planes of the two circles and therefore on the line of intersection of these two planes. Since, by 8 *d*, this line of intersection is the common tangent, it has no point other than the point of common tangency in common with either circle.

f. CONSTRUCTION: To construct the two great circles tangent to a given small circle and through a given point on the sphere, which point, together with its diametrically opposite point, is outside the given small circle.

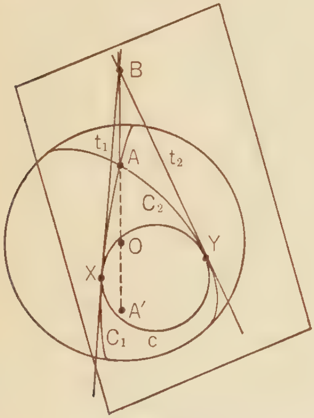

FIGURE 32

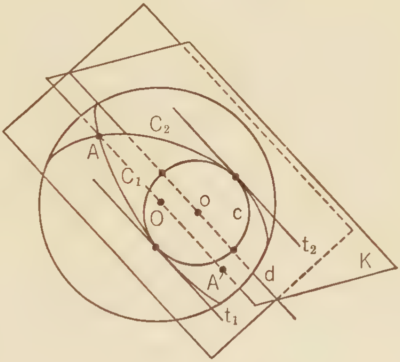

FIGURE 33

(1) Given, in Figure 32, the small circle *c* on the sphere, and the point *A*, also on the sphere. *A* and its diametrically opposite point *A′* are outside *c*. To construct the two great circles through *A* tangent to *c*:

(2) The extended diameter of the sphere through *A* and *A′* will either meet the plane of *c* in a point *B* outside the sphere, or this diameter will be parallel to the plane of *c*. Assume first the former case:

(3) From *B* draw in the plane of *c* the two tangents t_1, t_2 to *c*, having *X* and *Y*, respectively, as points of tangency. The required great circles, C_1 and C_2, are those in the central planes through t_1 and t_2, respectively.

(4) By construction, t_1 and t_2 are tangent to *c*, and therefore to the sphere at *X* and *Y*, respectively.

(5) Hence, t_1 and t_2 are tangent to C_1 and C_2, respectively, as the tangents

are in the planes of their respective great circles and have with these circles no other point in common besides X and Y, respectively.

(6) Then C_1 and C_2, having respective common tangents with c, are tangent to c. C_1 and C_2 pass through A, since, by construction, the central planes through t_1, t_2 are the planes determined by t_1, t_2, respectively, and AOA'.

(7) Assuming now that AA' is parallel to the plane of c (see Figure 33), pass the plane K through AA' and the center of c, cutting the plane of c in the diameter d of c, which is parallel to AA'. Then let t_1 and t_2 be the two tangents to c which are parallel to d. Then C_1 and C_2 are the great circles in the planes determined by AA' and t_1 and t_2, respectively, by reasoning similar to that for the first case.

g. THEOREM: A great circle tangent to a small circle is perpendicular to the great circle from the pole of the small circle to the point of tangency.

(1) In Figure 34, the great circle C_1 is tangent to the small circle c at point X and t is the common tangent at X to C_1 and c. C_2 is the great circle through X and P, the pole of c.

(2) Then the plane of $C_2 \perp$ the plane of c, by 4 *c*. Let oX be the intersection of these two planes.

(3) Since $t \perp Xo$, t is $\perp$ the plane of C_2, by 4 *e*.

(4) Since the plane of C_1 contains t, this plane of C_1 is $\perp$ the plane of C_2, by 4 *c*.

(5) Hence, $C_1 \perp C_2$.

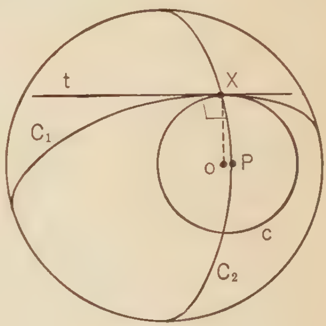

FIGURE 34

9. Spherical Triangles, Polar Triangles

a. DEFINITION: Any three-sided, closed, curvilinear figure on a sphere, bounded by minor arcs of three great circles (which do not intersect in the same pair of points), between consecutive points of intersection, is called a ***spherical triangle.***

In Figure 35, great circles C_1, C_2 intersect in the round dots, C_1, C_3 in the square dots, and C_2, C_3 in the triangular dots. Any curvilinear triangle, such as ABC, which has as vertices one dot of each kind and as sides a minor great-circular arc of each of the three great circles C_1, C_2, C_3 between these vertices is a spherical triangle.

NOTE: *Arcs of small circles cannot (by definition) serve as sides of a spherical triangle.* Figures so formed look like spherical triangles but are specifically excluded from this category by definition. Triangles whose sides are arcs of small circles can be dealt with, but not by methods applicable to spherical triangles.

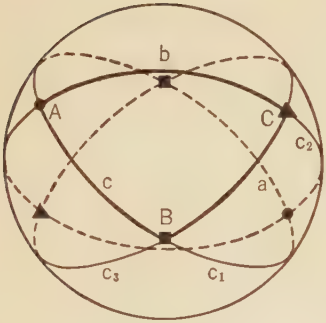

FIGURE 35

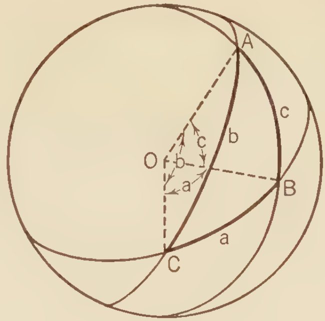

FIGURE 36

b. DEFINITIONS: The ***angles of a spherical triangle*** are spherical angles and are therefore measured in degrees of angle. The ***sides of a spherical triangle*** are minor arcs of great circles and are measured in degrees of arc. By joining the vertices ABC of the spherical triangle to the center O of the sphere there is formed the ***corresponding trihedral angle*** of the spherical triangle. (See Figure 36.) The ***vertex*** of the trihedral angle is at O and the three ***faces*** of the trihedral angle are the planes of the sides of the spherical triangle. Hence, the ***face angles*** of the trihedral angle (angles AOB, AOC, BOC) have the same measure as the corresponding sides of the triangle.

c. DEFINITION: Given a spherical triangle ABC. The great circle of side a divides the sphere into two hemispheres, in but one of which vertex A lies. Since the polar arcs of vertices B and C intersect at diametrically opposite points, only one of these two points of intersection can lie in that one of the two hemispheres determined by the side a in which vertex A lies. Call this point A'. Proceed in an analogous way to construct B' and C'. Then the spherical triangle $A'B'C'$ is called the ***polar triangle*** of the spherical triangle ABC. (See Figure 37.)

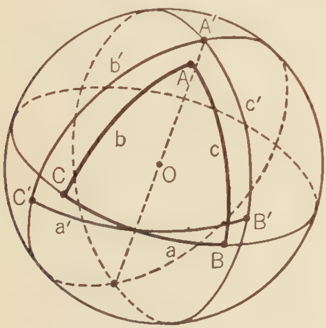

FIGURE 37

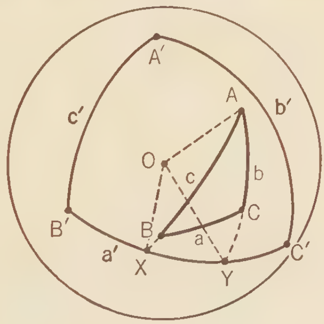

FIGURE 38

d. Theorem: If $A'B'C'$ is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$.

This follows from the above definition and theorem 6 *g*.

e. Definition: The two triangles of definition 9 *c* and theorem 9 *d* are said to be ***pole and polar*** and are conventionally labeled to suggest corresponding parts A, A'; b, b', etc. That is, A' is the pole of side a which is opposite vertex A; b' is the polar of vertex B which is opposite b; etc.

f. Theorem: A side of a spherical triangle is the supplement of the angle opposite the corresponding side in the polar triangle; an angle of a spherical triangle is the supplement of the side opposite the corresponding angle in the polar triangle; or

$$A + a' = 180°; \quad b + B' = 180°; \quad \text{etc.}$$

In Figure 38, sides c and b of triangle ABC are extended away from A until they meet side a' of the polar triangle $A'B'C'$ in X and Y, respectively. Then arc AX = arc AY = arc $B'Y$ = arc $C'X$ = 90°, by 6 *e*. Furthermore, arc XY has the same measure as angle A, by 6 *j*. Therefore, a' = arc $B'C'$ = arc $B'Y$ + arc XC' − arc XY = 90° + 90° − A = 180° − A.

g. Theorem: The sum of two sides of a spherical triangle always exceeds the third side; or $a + b > c$.

In Figure 39, the trihedral angle corresponding to the spherical triangle is drawn. The face angles of this trihedral angle have the same measures as the respective sides of the triangle. The theorem is therefore established by 5 *b*.

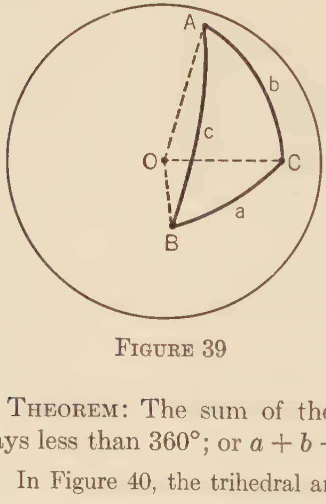

Figure 39

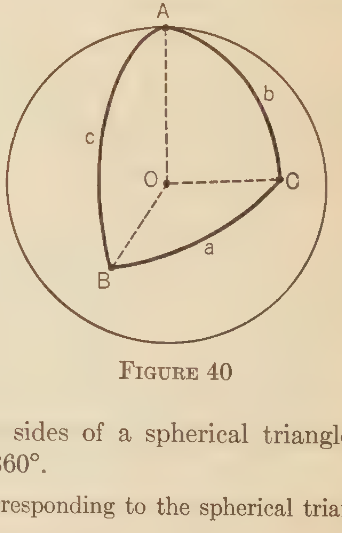

Figure 40

h. Theorem: The sum of the three sides of a spherical triangle is always less than 360°; or $a + b + c < 360°$.

In Figure 40, the trihedral angle corresponding to the spherical triangle is drawn. The face angles of this trihedral angle have the same measures as the respective sides of the triangle. The theorem is therefore established by 5 *c*.

i. THEOREM: The sum of the three angles of a spherical triangle is always greater than one and less than three straight angles; or $180° < A + B + C < 540°$.

From theorem 9 *f*, $A + B + C = 540° - (a' + b' + c')$
Since $a' + b' + c' > O$, $A + B + C < 540°$.
Since $a' + b' + c' < 360°$ by the above theorem, $A + B + C > 180°$.

j. THEOREM: The order of magnitude of the sides of a spherical triangle is the same as the order of magnitude of the corresponding angles opposite the sides; or,

$$\text{if } A < B < C, \text{ then } a < b < c,$$

and reciprocally,

$$\text{if } a < b < c, \quad \text{then } A < B < C.$$

The proof as given in spherical geometry follows from two applications of the

THEOREM: If two angles of a spherical triangle are unequal, the sides opposite the unequal angles are unequal, and unequal in the same sense.

The usual proof of this theorem is based upon theorems on isosceles triangles, which in turn are based on tedious theorems on congruent and symmetric spherical triangles. The proofs of the necessary theorems on isosceles spherical triangles are almost trivial in spherical trigonometry. Consequently, the student is referred to page 69 for the proofs necessary here. This inversion of order does not lead to any circular reasoning.

k. DEFINITION: Two spherical triangles will be said to be ***equal in area*** when they are either superposable as a whole or are the sum or same difference of corresponding triangles superposable in pairs.

l. DEFINITIONS: The ***plane triangle of a spherical triangle*** is the plane triangle of the vertices of the spherical triangle. The ***small circle of a spherical triangle*** is the small circle of the vertices of the spherical triangle. The ***pole of a spherical triangle*** is the pole of its small circle.

m. THEOREM: Two spherical triangles of respectively equal sides are equal in area.

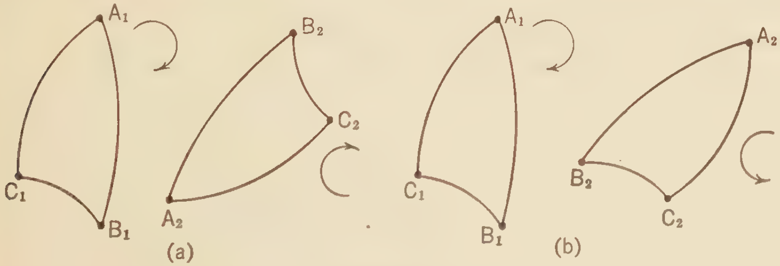

FIGURE 41

(1) Let one of the two spherical triangles be kept fixed and the other moved so that the planes of the two plane triangles of the spherical triangles are identical and so that the convexities of the two spherical triangles are the same.

(2) Two cases will be possible: *Case I:* The cyclic order of the corresponding vertices, as viewed from the convex sides, is the *same* in both spherical triangles. (See Figure 41 *a*.) *Case II:* The cyclic order of the corresponding vertices, as viewed from the convex sides, of one spherical triangle is *opposite* to this order in the other spherical triangle. (See Figure 41 *b*.)

(3) *Case I:* The plane triangle of one spherical triangle can be superposed on the plane triangle of the other (as they are congruent by s.s.s., since equal chords intercept equal arcs) *without a rotation out of its plane*, thus preserving the similar convexity of the two spherical triangles. Since a great circle is determined by two points on a sphere, the two spherical triangles will be superposed and, therefore, their areas are equal by definition. (See Figure 42 *a*.)

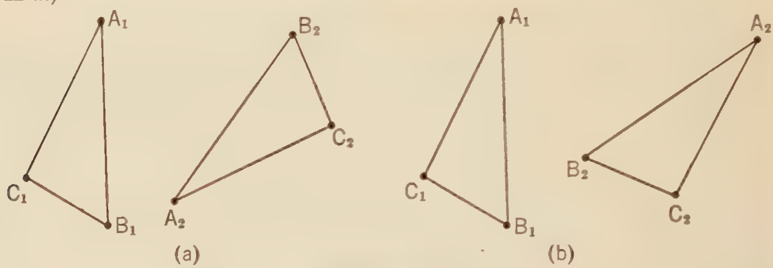

FIGURE 42

(4) *Case II:* (*a*) The plane triangle of one spherical triangle cannot be superposed on the plane triangle of the other *except by a rotation out of its plane*, which would destroy the similar convexity of the two spherical triangles and thereby make their superposition impossible. (See Figure 42 *b*.)

(*b*) The small circles of the two spherical triangles are equal, since they are the circumcircles of the congruent (s.s.s.) plane triangles of the spherical triangles.

(*c*) Hence, the great-circle distances from the poles, P_1, P_2, of the spherical triangles to their small circles, and therefore to the vertices of the spherical triangles, are all equal to one another. (See Figure 43.)

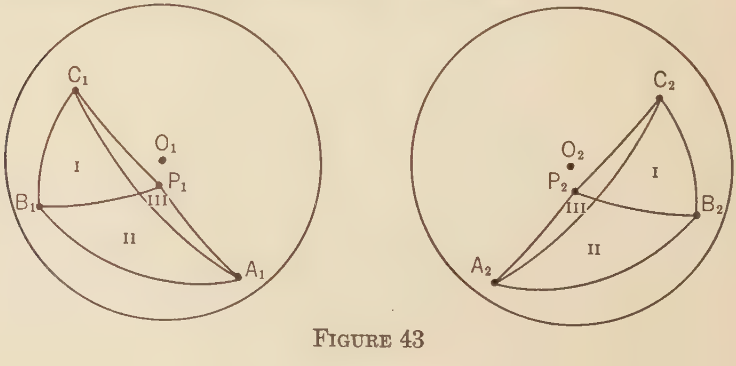

FIGURE 43

(*d*) These great-circle arcs from the poles of the spherical triangles to their vertices therefore divide each of the two spherical triangles into the sum or into the same difference of three isosceles spherical triangles. Furthermore, to each isosceles spherical triangle making up one of the given spherical triangles there is a corresponding isosceles spherical triangle of respectively equal sides making up the other spherical triangle. Consider the pair of corresponding isosceles plane triangles $B_1P_1C_1$ and $B_2P_2C_2$ in Figure 44.

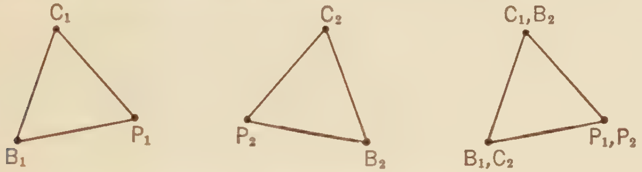

FIGURE 44

(*e*) The plane triangle $B_2P_2C_2$ can be superposed on the plane triangle $B_1P_1C_1$ *without a rotation out of the plane of* $B_2P_2C_2$, because these two congruent (s.s.s.) triangles are *isosceles*. Therefore, superposing the pairs of plane triangles of the isosceles spherical triangles need not alter the convexity of these isosceles spherical triangles (though superposing non-corresponding base vertices). Consequently, the component isosceles spherical triangles can be superposed in pairs.

(*f*) Therefore, the areas of the two given spherical triangles are equal by definition.

n. DEFINITION: The ***spherical excess,*** E, of a spherical triangle is the quantity $A + B + C - 180°$, where A, B, C are the measures of the angles of the spherical triangle ABC.

o. THEOREM: The ***area of a spherical triangle*** is that part of the area of the whole sphere (on which the spherical triangle is located) that the spherical excess of the spherical triangle is of 720°, or:

$$K = \frac{E}{720°} 4\,\pi r^2$$

(1) Extend the sides of the spherical triangle ABC (see Figure 45) to complete the great circles of the sides. Eight spherical triangles are thus formed: triangles ABC, CAB', $CB'A'$, and $A'CB$ are on that half of the sphere represented as being out from the paper, and the other four triangles are in the half of the sphere lying behind the plane of the paper.

(2) The spherical triangle $A'B'C$ (on the front half of the sphere) is equal in area to the spherical triangle ABC' (on the back half of the sphere) by the above theorem, since the sides of the

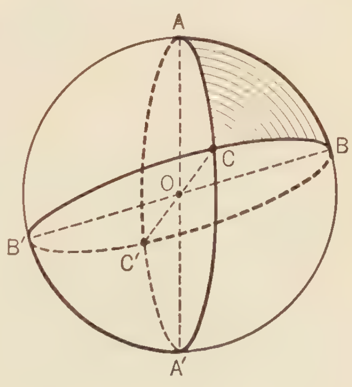

FIGURE 45

two triangles are respectively equal by the equality of the plane central angles at O.

(3) Hence, $\triangle ABC + \triangle A'B'C = \triangle ABC + \triangle ABC'$
$= \text{lune of angle } C$

(4) $\therefore$ lune of angle B + lune of angle A + lune of angle C = hemisphere + 2 ($\triangle ABC$).

$$(5)\ \therefore 2(\triangle ABC) = \left(\frac{B}{180°}\right)\text{hemisphere} + \left(\frac{A}{180°}\right)\text{hemisphere} + \left(\frac{C}{180°}\right)\text{hemisphere} - \text{hemisphere.}$$

$$\therefore 2(\triangle ABC) = \left(\frac{A+B+C-180°}{180°}\right)\text{hemisphere.}$$

$$\therefore \quad \triangle ABC = \left(\frac{E}{360°}\right)\text{hemisphere} = \left(\frac{E}{720°}\right)\text{sphere.}$$

B. DEFINITIONS AND FORMULAS FROM PLANE TRIGONOMETRY

10. Angles

a. Definition: When a ray (a straight line extending from a fixed point to infinity in but one direction) rotates about its finite end from an initial position to any final position, it is said to ***generate an angle*** whose ***vertex*** is the fixed finite end of the ray and whose ***sides*** are the initial and final positions of the ray. (See Figure 46.)

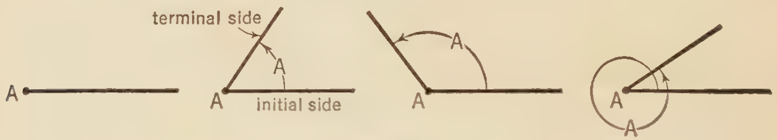

Figure 46

b. Convention as to Positive Angles: The positive direction of generation of an angle is taken as counter-clockwise.

c. Units in Angular Measure:

(1) *The Sexagesimal System.* In this system the angle formed by one complete revolution of the terminal line is considered divided into 360 equal (i.e., superposable) angles called ***degrees,*** each of which is subdivided into 60 equal ***minutes*** of 60 equal ***seconds.*** Hence, a ***right angle*** (that formed

when two lines intersect perpendicularly, or so as to form four equal angles) contains $\frac{360°}{4} = 90°$.

(2) *The Radian System.* In this system the unit angle, called the ***radian,*** is the angle at the center of the circle subtending an arc on the circumference of a circle equal to the radius of the circle. Since the circumference of a circle contains the radius 2π times, there are 2π radians in one complete circuit. Hence, $360° = 2\pi$ radians or

$$180° = \pi \text{ radians}. \tag{1}$$

When no unit symbol is shown for the measure of an angle, it is assumed to be expressed in radians.

The definition of radian immediately gives the relation

$$s = r\theta \tag{2}$$

where s = arc length (in units of the radius) along any circle of radius r, and θ is the angle in radians subtended by the arc s at the center of the circle. (See Figure 47.)

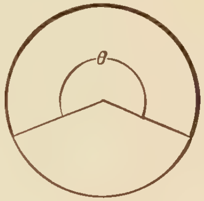

FIGURE 47

11. Definitions of the Trigonometric Functions of Angles

a. Given any angle A. Let the origin O of a Cartesian system of axes be at A and let the positive x-axis lie along the initial side of the angle. Let $P{:}(x, y)$ be any point on the terminal side of the angle. Call $OP = r$, considered as always positive, the ***distance.***

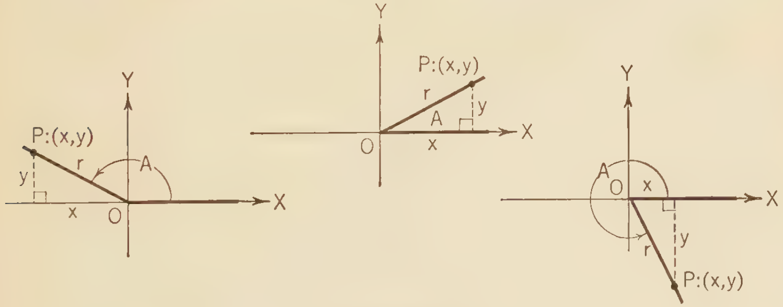

FIGURE 48

b. Then the sine, cosine, tangent, cotangent, secant, and cosecant functions of angle A are defined (using the conventional suggestive abbreviations for the names of these functions) as

$$\sin A = \frac{\text{ordinate}}{\text{distance}} = \frac{y}{r} \qquad \cos A = \frac{\text{abscissa}}{\text{distance}} = \frac{x}{r}$$

$$\tan A = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x} \qquad \cot A = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y} \tag{3}$$

$$\sec A = \frac{\text{distance}}{\text{abscissa}} = \frac{r}{x} \qquad \csc A = \frac{\text{distance}}{\text{ordinate}} = \frac{r}{y}$$

12. Variation of the Trigonometric Functions

a. From these definitions of the trigonometric functions it immediately follows that

(1) *The sine and cosine never numerically exceed unity.*
(2) *The secant and cosecant are never numerically less than unity.*
(3) *The tangent and cotangent can take on all values, plus or minus.*

b. Figure 49 shows the graphs of the trigonometric functions.

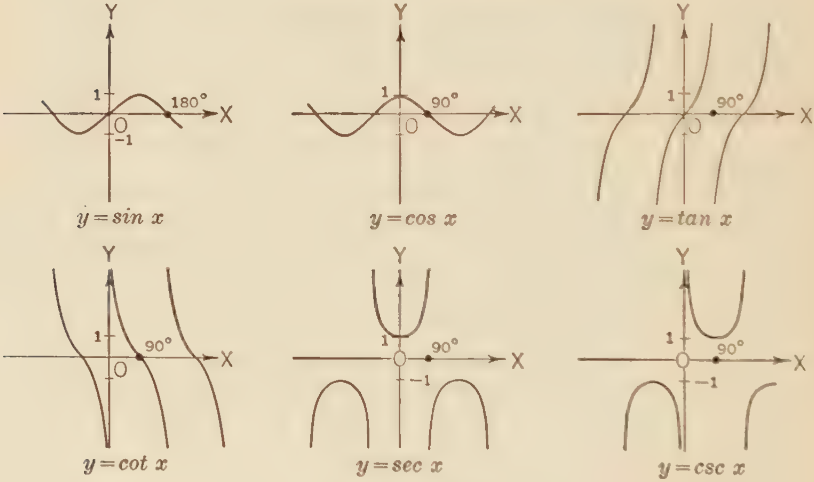

FIGURE 49

13. Signs of the Trigonometric Functions

a. Since r is always positive, the sign of a trigonometric function of an angle is given by the signs of x and y of the point P, which, in turn, are determined by the quadrant in which P on the terminal side of the angle lies.

b. DEFINITION: An angle is said to ***lie in a given quadrant*** if its terminal side lies in this quadrant when its initial side lies along the positive x-axis.

c. Accordingly, the signs of the trigonometric functions of the various quadrant angles are given by

First Quadrant x and y pos. All positive.

Second Quadrant $\left\{\begin{matrix} x \text{ neg.} \\ y \text{ pos.}\end{matrix}\right\}$ $\left\{\begin{matrix}\text{cosine, tangent} \\ \text{secant, cotangent}\end{matrix}\right\}$ neg.; $\left.\begin{matrix}\text{sine} \\ \text{cosecant}\end{matrix}\right\}$ pos.

Third Quadrant x and y neg. $\left\{\begin{matrix}\text{sine, cosine} \\ \text{cosecant, secant}\end{matrix}\right\}$ neg.; $\left.\begin{matrix}\text{tangent} \\ \text{cotangent}\end{matrix}\right\}$ pos.

Fourth Quadrant $\left\{\begin{matrix}x \text{ pos.} \\ y \text{ neg.}\end{matrix}\right\}$ $\left\{\begin{matrix}\text{sine, tangent} \\ \text{cosecant, cotangent}\end{matrix}\right\}$ neg.; $\left.\begin{matrix}\text{cosine} \\ \text{secant}\end{matrix}\right\}$ pos.

14. Fundamental Relations between the Trigonometric Functions

a. RECIPROCAL RELATIONS: From the definitions of the trigonometric functions it follows that

(1) *The sine and cosecant are reciprocal functions.*
(2) *The tangent and cotangent are reciprocal functions.*
(3) *The cosine and secant are reciprocal functions.*

b. SQUARED RELATIONS: From the Pythagorean relation for right triangles it follows that

$$\sin^2 A + \cos^2 A = 1 \tag{4}$$
$$\sec^2 A - \tan^2 A = 1 \tag{5}$$
$$\csc^2 A - \cot^2 A = 1 \tag{6}$$

c. $\tan A = \dfrac{\sin A}{\cos A}$

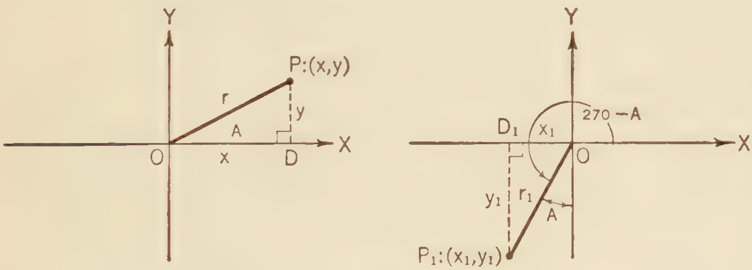

FIGURE 50

d. FUNCTIONS OF $(n\,90° \pm A)$: Any trigonometric function of an angle $(n\,90° \pm A)$ is numerically equal to the corresponding function of A, if n is an even integer, and to the corresponding cofunction of A, if n is an odd integer. Whether there is a change in sign in this relation will depend upon the quadrants of the two angles $(n\,90° \pm A)$ and A.

Figure 50 shows the case for A acute, $n = 3$, and the minus sign and indicates the method of proof for all cases: Construct $OP_1 = OP$. Then $\triangle OPD \cong \triangle OP_1D_1$. Therefore, $x_1 = -y$, $y_1 = -x$, $r_1 = r$. Applying the definitions of the trigonometric functions of the angle $270° - A$ and of the angle A proves the relation for this case.

15. Functions of Special Angles

a. The 30°, 60° *multiples:* From the fact that an altitude of an equilateral triangle is also an angle bisector and a median, all functions of 30°, 60°, and

$n\,90° \pm$ either of these two angles are expressible exactly. Figure 51 indicates how these values are to be found (e.g., $\cos 150° = -\frac{1}{2}\sqrt{3}$, etc.).

b. The 45° *multiples:* From the isosceles right triangle all functions of 45° and $n\,90° \pm 45°$ are expressible exactly. Figure 52 indicates how these values are to be found (e.g., $\tan 135° = -1$, etc.).

c. The 90° *multiples:* All functions of any angle whose terminal side lies on one half of one of the co-ordinate axes are expressible exactly by observing that in these cases one co-ordinate of the point P on the terminal side of the angle becomes zero while the other becomes numerically equal to the distance r. Figure 53 indicates how these values are to be found (e.g., cot 180° is undefined, approaching either plus or minus infinity as the angle approaches 180°; $\cos 180° = -1$; $\sin 180° = 0$).

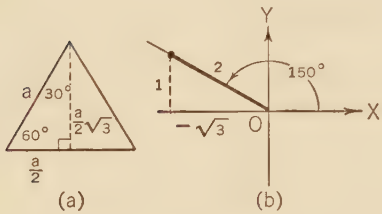

FIGURE 51

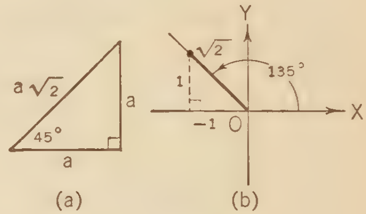

FIGURE 52

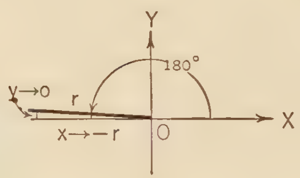

FIGURE 53

16. Functions of General Angles

The ratios which express the approximate values of the trigonometric functions of general angles are tabulated in tables of 4, 5, or more places of decimals. A less accurate, but frequently adequate, approximation can be obtained from a slide rule equipped with trigonometric scales.

17. Trigonometric Functions Determined from a Given Function

Because of the dependence of the trigonometric functions on one another, any one function of an angle determines all the others, provided also the quadrant of the angle is known. The method of procedure in each case is to:

Draw an angle fitting the data, which are to be shown on the figure; evaluate a third side of a right triangle by the Pythagorean relation; read from the figure the desired trigonometric functions. Figure 54 indicates this procedure for two special cases:

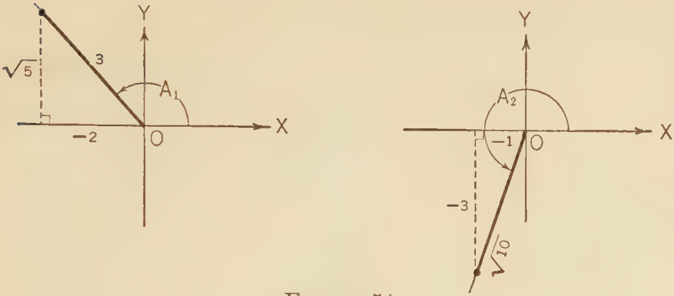

FIGURE 54

EXAMPLE 1: $A_1 = \cos^{-1}(-\frac{2}{3})$,* A_1 in II.

$$\therefore \sin A_1 = \frac{\sqrt{5}}{3};\ \cot A_1 = \frac{-2}{\sqrt{5}};\ \text{etc.}$$

EXAMPLE 2: $A_2 = \tan^{-1} 3$,* $\csc A_2$ negative.

$$\therefore \sin A_2 = \frac{-3}{\sqrt{10}};\ \sec A_2 = -\sqrt{10};\ \text{etc.}$$

18. The Addition Formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad (7)$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad (8)$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad (9)$$

Figure 55 for the case of A and B acute and $A + B$ obtuse should suggest the method of proof in the case of $\sin(A + B)$ and $\cos(A + B)$. Similar figures can be drawn for other combinations of A, B, and $A + B$.

The $\tan(A + B)$ formula is obtained by dividing $\sin(A + B)$ by $\cos(A + B)$ and then dividing numerator and denominator of the resulting fraction by an expression which will yield tangents of angles A and B.

The formulas for the negative signs are obtained by replacing B by $-B$ and using 14 d for $n = 0$ and the minus sign.

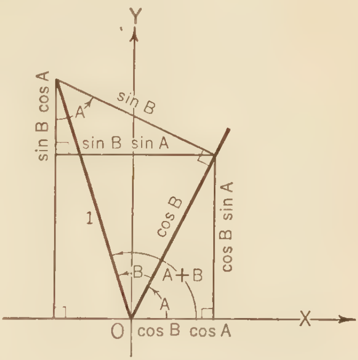

FIGURE 55

19. The Double-Angle Formulas

By replacing B by A in the addition formulas the following result:

$$\sin 2A = 2 \sin A \cos A \qquad (10)$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 \qquad (11)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \qquad (12)$$

* $\cos^{-1}(-\frac{2}{3})$ or "arc cosine $(-\frac{2}{3})$" is an *angle* whose cosine equals $-\frac{2}{3}$.

20. The Half-Angle Formulas

$$\sin A/2 = \pm\sqrt{\frac{1-\cos A}{2}} \tag{13}$$

$$\cos A/2 = \pm\sqrt{\frac{1+\cos A}{2}} \tag{14}$$

$$\tan A/2 = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A} \tag{15}$$

The $\pm$ sign is to be determined in each case from the quadrant of the angle $A/2$.

The first two half-angle formulas are derived by solving, respectively, the second and third formulas in (**11**) for the function of the single angle in terms of the double angle. Replacing the double angle by a single angle, and therefore the single angle by the half-angle, yields the formulas for the sine and cosine of the half-angle.

The third formula is derived by division of the first two and then rationalizing by conjugate multiplication.

21. The Factor Formulas

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \tag{16}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \tag{17}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \tag{18}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \tag{19}$$

These are derived by adding or subtracting appropriate pairs of formulas in **18**. If, for instance, the two formulas (7) are added, there results $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

Solving for A and B in terms of $(A+B)$ and $(A-B)$ yields the first of the above formulas.

22. The Triangle Laws

These are laws applying to any plane triangle of angles A, B, C and corresponding opposite sides a, b, c.

a. The Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The proof is accomplished by dropping altitudes onto two sides and equating two expressions for each altitude from the definitions of the trigonometric functions. (Each altitude, forming two right triangles, makes the definitions of trigonometric functions immediately applicable.) (See Figure 56: $h_c = b \sin A = a \sin B$, etc.)

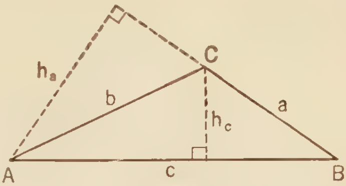

FIGURE 56

b. The Law of Cosines:

$$a^2 = b^2 + c^2 - 2\,bc \cos A, \text{ etc.}$$

This law expresses the square of an assumed unknown side in terms of the assumed known other two sides and the included angle. It is proved by dropping an altitude onto one of the known sides, forming segments ϕ_1 and ϕ_2. The square of the unknown side is then expressed by the Pythagorean Theorem in terms of the altitude and one of the ϕ's, which latter is then written in terms of a known side and the other ϕ. Another application of the Pythagorean Theorem eliminates the squares of the remaining auxiliaries (ϕ and the altitude) in favor of the other known side. Finally the definition of the cosine eliminates the remaining auxiliary. (See Figure 57.)

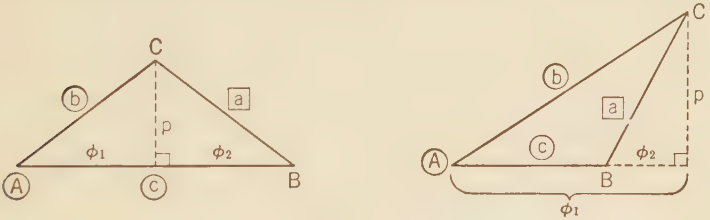

FIGURE 57

$$a^2 = p^2 + \phi_2^2\ ;\ \phi_2 = \pm\,(c - \phi_1)$$
$$a^2 = p^2 + c^2 - 2\,c\,\phi_1 + \phi_1^2$$
$$a^2 = b^2 + c^2 - 2\,c\,(b \cos A), \text{ etc.}$$

c. The Half-Angle Law:

$$\tan\frac{A}{2} = \frac{r}{s-a};\ \tan\frac{B}{2} = \frac{r}{s-b};\ \tan\frac{C}{2} = \frac{r}{s-c}$$

where $s = \frac{1}{2}(a + b + c)$ and $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$

This law is merely a devious algebraic transformation of the law of cosines by means of the half-angle formulas to yield formulas which, because they involve products instead of sums and differences, will be more suitable for logarithmic solutions of angles of triangles given the three sides.

Since an angle is desired, it is natural to begin by solving a cosine formula for the angle:

$$a^2 = b^2 + c^2 - 2\,bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}$$

The terms b^2, c^2, $2\,bc$ suggest $(b \pm c)^2$, which can be introduced by respectively adding or subtracting cos A from 1. Exploring this lead gives:

$$1 + \cos A = \frac{b^2 + 2\,bc + c^2 - a^2}{2\,bc}$$

$$1 - \cos A = \frac{a^2 - b^2 + 2\,bc - c^2}{2\,bc}$$

Algebraic factoring then reduces the right-hand sides to products:

$$1 + \cos A = \frac{(b + c)^2 - a^2}{2\,bc} = \frac{(b + c - a)\,(b + c + a)}{2\,bc} = \frac{2\,(s - a)\,2\,s}{2\,bc}$$

$$1 - \cos A = \frac{a^2 - (b - c)^2}{2\,bc} = \frac{(a - b + c)\,(a + b - c)}{2\,bc} = \frac{2\,(s - b)\,2\,(s - c)}{2\,bc}$$

The half-angle formulas then give:

$$\cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}} \qquad \sin \frac{A}{2} = \sqrt{\frac{(s - b)\,(s - c)}{bc}}$$

and division of the second by the first gives the above formulas to be derived.

d. The Law of Tangents:

$$\frac{\tan \frac{1}{2}\,(A - B)}{\tan \frac{1}{2}\,(A + B)} = \frac{a - b}{a + b}, \text{ etc.}$$

Since the sum of the angles of a plane triangle is constant, any one angle determines the sum of the other two. The individual angles of this sum could then be found if their difference could be computed. This reasoning may suggest exploring functions of the sum and difference of two angles. Since the factor formulas involve just such functions, expressing the sums or differences of the sines or cosines of two angles is suggested. The law of sines is the point of departure:

Writing the sine law

$$\frac{\sin A}{\sin B} = \frac{a}{b}$$

and taking this proportion in subtraction and addition, one obtains:

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a - b}{a + b}.$$

Applying the factor formulas (**16**) and (**17**) gives

$$\frac{2 \cos \frac{1}{2}\,(A + B) \sin \frac{1}{2}\,(A - B)}{2 \sin \frac{1}{2}\,(A + B) \cos \frac{1}{2}\,(A - B)} = \frac{a - b}{a + b} = \cot \tfrac{1}{2}\,(A + B) \tan \tfrac{1}{2}\,(A - B)$$

from which the desired law is obvious.

CHAPTER 1

Fundamental Concepts

1. Purpose and Scope of Spherical Trigonometry

Spherical trigonometry is to the surface of a sphere what plane trigonometry is to a plane. Since we live on the surface of a sphere and imagine the heavenly bodies as moving in a celestial sphere about the earth, the applications of spherical trigonometry are many and obvious. As long as relatively short distances are considered, plane trigonometry is entirely adequate. When, however, we are concerned with flying or sailing more than a few hundred miles, we must take the curvature of the earth's surface into account; i.e., we must use spherical trigonometry. Furthermore, calculations involving the heavenly bodies, by which time is measured and positions on the earth's surface are determined, make use of the concept of a spherical shell about the earth as center. Upon this celestial sphere all the heavenly bodies are imagined to be projected. Measurements based on the positions of these heavenly bodies must necessarily use spherical trigonometry.

The student will find it interesting and instructive to watch for similarities and differences between analogous ideas in spherical trigonometry and plane trigonometry. In this way — by comparison and contrast — he will fortify his knowledge of plane trigonometry while exploring spherical trigonometry. Some of these analogues are striking; others are hardly perceptible.

2. Geodesics or Lines of Minimum Distances

Just as in plane trigonometry the plane triangle is fundamental (because it is the polygon of the smallest number of sides and because it is rigid; i.e., determined when its three sides are given), so in spherical trigonometry the ***spherical triangle*** is the fundamental object of consideration. Sides of plane triangles are ***geodesics*** (lines of shortest distance between pairs of points on them) in the plane. Since the idea of shortest distance between two points is obviously an efficient one, it is desirable to make the sides of spherical triangles by definition geodesics on the sphere. The information necessary for such a definition is to be found in the following definitions and theorem.

DEFINITION: *A* **great circle** *on a sphere is the circle of intersection of the surface of the sphere and a plane through the center of the sphere.*

DEFINITION: *A* **small circle** *on a sphere is the circle of intersection of the surface of the sphere and a plane not through the center of the sphere.*

THEOREM: *The geodesic or shortest distance on the surface of a sphere between two given points (not diametrically opposite) on this surface is the minor arc of the unique great circle through the two given points.*

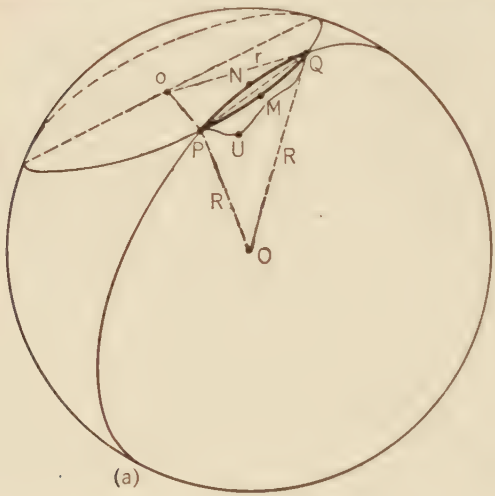

FIGURE 58

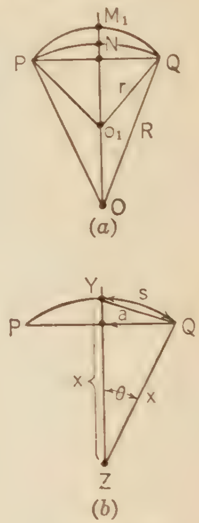

FIGURE 59

1. In Figure 58 P and Q are the two given points on the surface of the sphere. PNQ is the minor great-circle arc between P and Q, with N as its midpoint, O, the center of the sphere, as its center, and R, the radius of the sphere, as its radius. PMQ is the minor arc of some small circle through P and Q, with M as its midpoint, o as its center, and r as its radius. PUQ is any other arc on the sphere between P and Q; that is, PUQ is not a plane arc.

2. It will be assumed to be intuitively obvious that PUQ, an arc of a *space* curve between P and Q, will be longer than the shortest arc of a *plane* curve between these two points. Hence it remains to show that PMQ is longer than PNQ. That PNQ is unique follows from the fact that the three points P, Q, and O determine but one plane.

3. Imagine the plane figure $oPMQ$ rotated about the chord PQ until the plane of this figure coincides with the plane of $OPNQ$ with o lying

between the chord and O. Figure 59 *a* shows the result of this rotation with M_1, o_1 indicating the new positions of M and o, respectively. Since the sum of two sides of a plane triangle exceeds the third,

$$Oo_1 + o_1Q = OM_1 > OQ = ON.$$

Therefore M_1 and the whole of arc PM_1Q except the end points lie above the sphere. Consequently, if a string were fitted over the arc PNQ, it is at least intuitively obvious that to be made to fit over PM_1Q or over its equal PMQ, this string would have to be stretched. Therefore, it is at least intuitively obvious that the arc PNQ is less than the arc PMQ.

4. For a more rigorous proof that

$$\text{arc } PM_1Q > \text{arc } PNQ,$$

see Figure 59 *b* and investigate the variation in circular arcs subtended by the chord PQ as the position of the center of such arcs, and consequently also the radii of these arcs, vary. Letting a be half the chord PQ, s half the subtended varying arc of varying radius x, and $2\,\theta$ the varying central angle, we have

$$s = x\,\theta,\ \theta \text{ in radians; } x = a \csc \theta.$$

(See Introduction 10 *c* (2) and 11 *b*.)

$$s = a\,\theta \csc \theta = a\,\frac{\theta}{\sin \theta}.$$

But

$$\text{area sector } ZQY = \tfrac{1}{2} x^2\,\theta > \text{area triangle } ZQY = \tfrac{1}{2} x\,(x \sin \theta).$$

Therefore, $\theta > \sin \theta$ and $\dfrac{\theta}{\sin \theta} > 1$, which shows that s is certainly greater than a. Comparison of tables for θ in radians with tables for $\sin \theta$ will quickly show that as θ increases, the ratio of θ to $\sin \theta$ numerically increases.* For example,

$$\theta = \frac{\pi}{6}: \frac{\theta}{\sin \theta} = \frac{0.5236}{0.5000} = 1.047$$

$$\theta = \frac{\pi}{4}: \frac{\theta}{\sin \theta} = \frac{0.7854}{0.7071} = 1.111$$

$$\theta = \frac{\pi}{2}: \frac{\theta}{\sin \theta} = \frac{1.5708}{1.0000} = 1.571.$$

Consequently, s increases as θ increases, or s decreases as θ decreases and as x increases. But the largest value possible for x is R, the radius of the sphere. Hence, the smallest value of s is that for which the arc lies along a great circle.

* If calculus is used, we have

$$s = a\,\frac{\theta}{\sin \theta} = a\,\theta \csc \theta,$$

$$\frac{ds}{d\theta} = a\,\frac{\csc \theta}{\tan \theta}\,(\tan \theta - \theta).$$

But for first quadrant θ, $\tan \theta > \theta$, and $\csc \theta$ and $\tan \theta$ are positive. Hence, for first quadrant θ (those values of θ with which we are concerned), $\dfrac{ds}{d\theta} > 0$, and s is an increasing function of θ.

3. Spherical Triangles

Having seen in the previous section what arcs on the sphere are geodesics, we are in a position to define spherical triangles as follows:

DEFINITION: ***Spherical triangles*** *are closed figures formed on the surface of any sphere by arcs of three great circles, each such arc being less than a half-circle.*

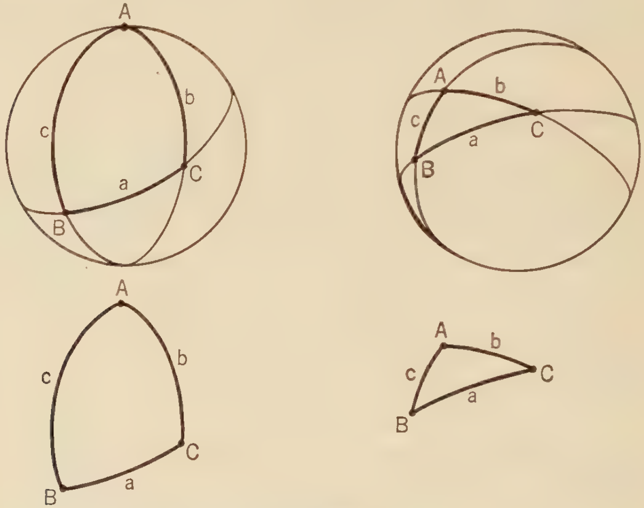

FIGURE 60

The arcs of the great circles are the ***sides*** and the points of intersection of the arcs are the ***vertices*** of the spherical triangles.

Spherical triangles are labeled as are plane triangles in general: Large letters, usually A, B, and C, for the vertices or the measure of the angles at these vertices, and corresponding small letters for the opposite sides or their measures. (See Figure 60 for examples.)

It is occasionally desirable to consider on a sphere certain triangles one or more of whose sides are arcs of *small* circles. In Figure 61, P_NADB is such a triangle on the surface of the earth, where the side ADB is an arc of a parallel of latitude. (Arc ACB is the great-circle arc between A and B.) As will be pointed out in Chapter 4, such triangles can be measured

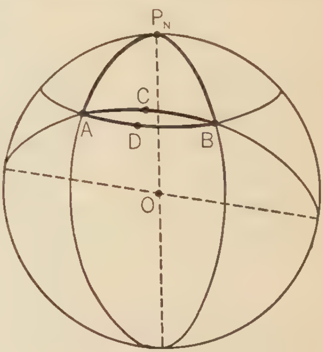

FIGURE 61

but are measured by methods other than those applying to spherical triangles, from which class of triangles the triangle P_NADB is, by definition, excluded.

4. Fundamental Concepts of Plane and Spherical Trigonometry Compared

From the foregoing it is apparent that arcs of great circles are to be considered the analogues in spherical trigonometry of straight lines in plane trigonometry. Each is the geodesic (shortest distance between two points) in its domain, and hence, by definition, each is used as a side of a triangle in its domain. A brief consideration of some of the more obvious differences between "straight lines" or geodesics in plane and in spherical trigonometry is instructive.

1. *All "straight lines"* (i.e., great circles) *in spherical trigonometry intersect one another in two points; hence, there are no parallel lines in spherical trigonometry.* This follows because all great circles intersect all other great circles, as their planes must intersect one another, since they must all contain the center of the sphere. (Cf. Introduction, 6 *c*.)

DEFINITION: *The* ***angle between two great-circular arcs*** *is the plane angle between the tangents to the two arcs at their point of intersection.*

In Figure 62, AC_1, AC_2 are two great-circular arcs intersecting at A. At_1, At_2 are the tangents at A to C_1 and C_2, respectively. By definition, the plane angle t_1At_2 is the angle between the two great-circular arcs. By drawing the radius AO it is seen that this angle is also the plane angle of the dihedral angle $C_1–AO–C_2$ formed by the two planes of the great circles AC_1, AC_2 intersecting in the radius OA. (Cf. Introduction, 4 *a*, *b*.) Hence, to say that the angle between two great-circular arcs is 60° is to say that their planes intersect in an angle of 60°, and two great-circular arcs are perpendicular when their planes are perpendicular.

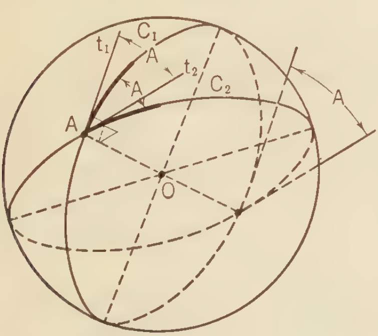

FIGURE 62

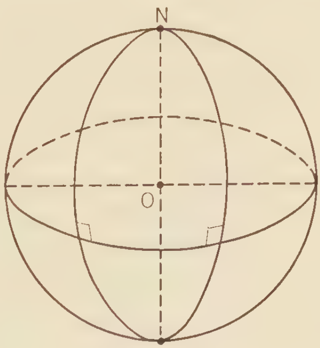

FIGURE 63

2. *Two "straight lines," each perpendicular to a third straight line, can intersect one another at any angle.* From 1 they cannot be parallel. By taking any two meridians on the earth as the two given straight lines and the equator as the third straight line (see Figure 63), the truth of this becomes apparent. All meridians are perpendicular to the equator, because their planes are perpendicular to the plane of the equator, since the axis of the earth is the diameter perpendicular to the plane of the equator. (See Introduction, 4 *c*.) Consequently,

3. *A spherical triangle can have one, two, or three right angles.* Just as we agree to restrict the size of the sides of spherical triangles to arcs less than half-circles, we also restrict their angles to less than straight angles. More generally we know that

4. *The sum of the angles of a spherical triangle is greater than one and less than three straight angles.* The proof of this, though not essential here, can be recalled by referring to Introduction, 9 *i*.

Whereas the length of a side of a plane triangle depends merely upon the distance between the vertices on this side, the length of a side of a spherical triangle must depend on the size of the sphere on which it is a triangle. Figure 64 shows two spherical triangles, ABC and $A'B'C'$, one on each of two concentric spheres, with the vertices of the triangle on the larger sphere projections, from the center of the two spheres, of the corresponding vertices of the triangle on the smaller sphere. We would naturally call these two triangles similar. The angles at the vertices of one triangle are equal respectively to the angles at the corresponding vertices of the other triangle. Furthermore, the *angular* measures of corresponding sides of the two triangles are equal, being the angular measures between radii to the pairs of vertices. The *linear* measures, however, of the sides of one triangle are not equal to the linear measures of the corresponding sides in the other. Consequently, we conclude that on a given sphere all the properties of a spherical triangle will be completely known when the angular measures of the angles of the triangle and the angular measures of its sides are known or can be deduced.

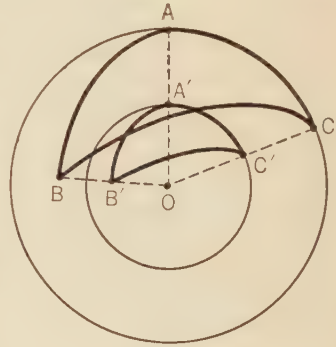

FIGURE 64

5. *In spherical triangles we shall consider the sides (as well as the angles) measured in* **degrees**; *i.e., in degrees of arc.* When the linear measure of

a side of a spherical triangle is desired, it can be immediately computed by the well-known formula $s = r\theta$, or "the linear measure of a circular arc equals the linear measure of the radius of the circle times the angular measure of the arc in radians." To facilitate this last computation in the case of spherical triangles on the earth's surface, the following convention has been adopted:

DEFINITION: *A* ***nautical mile*** *is the distance on the earth's surface covered by one minute of arc of a great circle.*

Consequently, the linear measure in nautical miles of a side of a spherical triangle on the earth's surface is equal to its measure in minutes of arc. To convert to land miles (if this should ever be required) the following relations can be used:

$$1 \text{ nautical mile} = 6080 \text{ feet}$$
$$1 \text{ land mile} = 5280 \text{ feet}$$

5. Suggestions for Sketching Spherical Triangles

Even though a student may learn how to perform mechanically the computation involved in solutions of spherical triangles, he will never really know what he is doing unless he can visualize the geometry involved by drawing simple sketches. A few suggestions toward this end are pertinent here.

1. Visualize a required spherical triangle as actually lying on a sphere represented by a large circle. Then, when its general aspects are thus made clear, it may be sufficient to consider the triangle by itself without the sphere.

2. Draw all great circles on a sphere (except that one in the plane of the paper) as ellipses with major axes along diameters of the sphere.

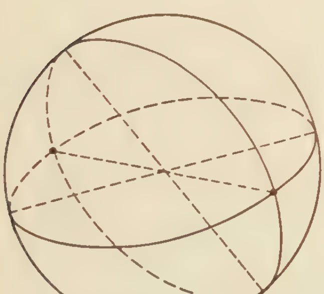

FIGURE 65

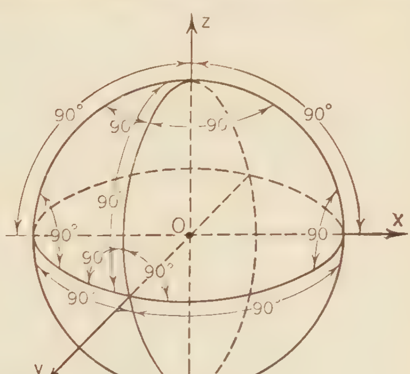

FIGURE 66

Draw the forward half with a full line and the half on the back of the sphere with a dotted line. Above all, remember that the two points where these two half-ellipses join must always be diametrically opposite. To make sure of this, show the major axis of the ellipse as a dotted diameter of the sphere as is indicated in Figure 65. Points of intersection of two great circles must be shown as being diametrically opposite one another.

3. Adopt a reasonable convention for the amount of perspective to be shown. Figure 66 shows a sphere with three mutually perpendicular great circles. The amount of baying out of the ellipses to represent the horizontal great circle (the equator, if the sphere is thought of as the earth) and the vertical great circle in the yz plane is arbitrary. But if the student has in his own mind some reasonably fixed convention as to the amount of baying out for these two great circles, he will be better able to visualize two great circles intersecting at any desired angle. Figure 67 illustrates this last point. All sketches should be freehand and quickly executed. All that is needed is a ready method of approximating the required geometrical relations.

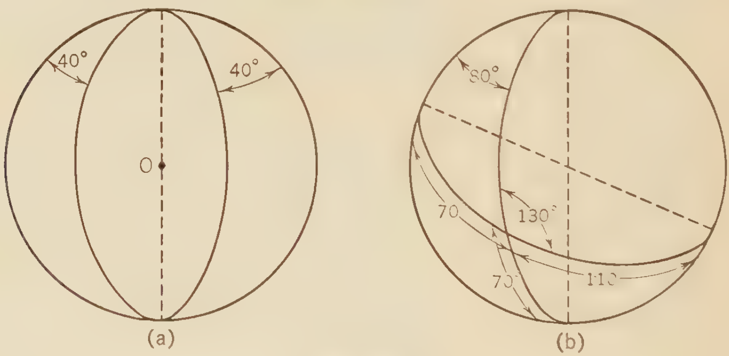

FIGURE 67

4. To draw the great circle through a given point on the sphere and perpendicular to a given great circle, first sketch the position of a pole of the great circle.

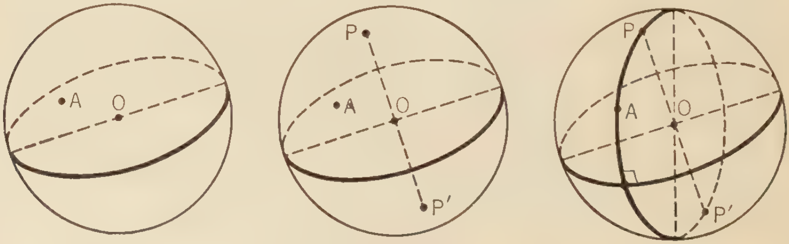

FIGURE 68

DEFINITION: *A* ***pole of a great circle*** *is either one of the two points on the surface of the sphere at which that diameter of the sphere which is perpendicular to the plane of the great circle pierces the surface of the sphere.* (Cf. Introduction, 6 *d*, ff.)

Then draw the great circle through the given point and this pole. (Cf. Introduction, 6 *f*.) Figure 68 illustrates this last point.

6. Problems on Chapter 1

1. Represent a sphere by a large circle G in the plane of the paper. Let A be a point on the sphere and place A on G 60° below and to the right of the top of G. Then:

Through A draw three great circles, G_1, G_2, G_3, represented as making with G respective angles of 90°, 45°, 20°. Curve G_1 so that it does not appear directly in front of the observer. Mark the angles of intersection of G_1, G_2, G_3 with G.

2. Represent a sphere by a large circle G in the plane of the paper. Let A be a point on the sphere not on G. Let A appear to be one third of the radius of G in from the circumference of G. Then:

(a) Through A draw a great circle G_1 represented as intersecting G at an angle of 20°.

(b) Through A draw a great circle G_2 represented as intersecting G_1 at an angle of 30°.

(c) Locate the poles P_1, P_1' of G_1 and the poles P_2, P_2' of G_2.

(d) Draw the great circle $P_1P_2P_1'P_2'$ and on it mark a 90° arc from P_1 to G_1 and a 90° arc from P_2 to G_2.

(e) Locate the poles of P_1P_2.

3. Represent a sphere by a large circle G in the plane of the paper. Let A be a point on the sphere, but not on G. Then:

(a) Draw a small circle g such that every point on g appears to be 30° of arc from A.

(b) Let B, C be two points on this small circle g, but not at ends of the same diameter of g.

(c) Draw the great circle through B and C and label the great-circle distance and the small-circle distance between B and C.

(d) Draw another small circle through B and C.

4. Draw five large figures each representing a sphere by a large circle G in the plane of the paper. Let A be the top of G in each figure. Then in each figure draw a spherical triangle with one vertex at A such that

(a) In the first figure the triangle is isosceles with the equal legs 90° and the vertex angle 20° at A.

(b) In the second figure one side through A is 170°, the side opposite A is 15°, and the angle at A is 150°.

(c) In the third figure angle A is 10° and the legs through A are 160° and 15°, respectively.

(d) In the fourth figure the triangle is equiangular with all angles 90°.

(e) In the fifth figure the triangle is equilateral with all sides 15°.

5. Draw three large figures, each representing a sphere by a large circle G in the plane of the paper. Let A be a point on the sphere, but not on G. Let

A appear to be one third of the radius of G in from the circumference of G. Then in each figure draw a triangle with A as one vertex such that

(a) In the first figure angle A is 20° and the sides through A are 100° and 30°, respectively.

(b) In the second figure the triangle is isosceles with the vertex angle A 170° and the opposite side 150°.

(c) In the third figure the triangle has two 90° angles, one of them at A, and the included side 30°.

6. On a large sketch show two points on the earth's surface, both on the 60° north parallel of latitude and differing in longitude by 180°. Mark the arc representing the distance between these two points along the parallel of latitude and also along the great circle through them. In terms of R, the radius of the earth, express the distance saved in taking this great-circle path between the two points instead of the parallel of latitude path. What approximately does this saving amount to, assuming the earth's radius is 4000 miles?

7. Follow directions in Problem 6 for two points on the 30° south parallel of latitude.

8. If the great-circle distance between two points having the same latitude on the earth's surface is the distance along the parallel of latitude, where must the two points be? What can be said of two points on the earth's surface such that the great-circle distance between them lies on a great circle through the poles of the earth?

CHAPTER 2

Right Spherical Triangles

7. Definition and Importance of Right Spherical Triangles

Right triangles play the same rôle in spherical trigonometry that they do in plane trigonometry. They are more simply solved than oblique triangles and they can be used effectively to divide up oblique triangles for solution.

DEFINITION: *A* ***right spherical triangle*** *is a spherical triangle with at least one right angle.* (The right angle will ordinarily be labeled C.)

To "solve" any spherical triangle is to find the angular measure of each unknown angle and each unknown side, given a certain set of known parts. From what follows it will be seen that, in general, the same number of parts are needed to solve spherical as plane triangles, namely, three. The exception in plane triangles of no solution if three angles only are given, will be seen later not to be an exception in spherical triangles.

In the case of right spherical triangles, then, "solving" will mean, given any two parts in addition to the assumed right angle, to compute the angular measures of the three remaining parts. When one of these two given parts, in addition to the already assumed right angle, is another right angle, the right spherical triangle becomes so specialized as to demand individual attention. Since fundamental solid geometry theorems are directly applicable to such particular right spherical triangles (as will later be pointed out), their special treatment is not difficult. This special treatment is necessary because the formulas to be derived for other right spherical triangles will be found to be inapplicable to the specialized right triangles.

DEFINITION: *A* ***general right spherical triangle*** *is a spherical triangle containing one and only one right angle.*

DEFINITION: *A* ***special right spherical triangle*** *is a spherical triangle containing at least two right angles.*

The solution of the general case is logically considered first to show the need for the special case.

To provide a method for solving general right spherical triangles, formulas, called ***Napier's Rules,*** will be derived and arranged so as to be easily remembered. The derivation of these right-triangle formulas draws heavily on theorems from solid geometry and is the most complicated part of spherical trigonometry as well as the basis for the solution of all spherical triangles.

8. Derivation of Formulas for Solving General Right Spherical Triangles

1. *Given* the general right spherical triangle ABC with $C = 90°$.

To derive formulas expressing each of the other five parts in terms of some other two of these five parts.

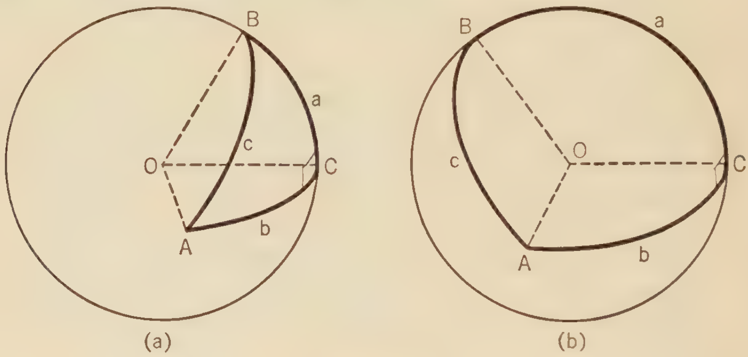

FIGURE 69

2. *Construction: a.* Connect the vertices A, B, and C in Figure 69 with O, the center of the sphere on which ABC is a right spherical triangle.

DEFINITIONS: In each case the figure thus formed by the planes of the three given sides of the right triangle, intersecting at the center, O, of the sphere, is called the ***trihedral angle***, corresponding to the spherical triangle. The point O is called the ***vertex*** of the trihedral angle. The planes OAB, OAC, OBC of the sides of the triangle are called the ***faces,*** the plane angles at the vertex are called the ***face angles,*** and the radii OA, OB, and OC are called the ***edges*** of this corresponding trihedral angle. (Cf. Introduction, 5 *a* and 9 *b*.)

b. Through B construct the plane perpendicular to OA.

That is, imagine this construction performed according to Introduction, 3 *d*. Several essentially different figures are possible, depending upon the position of this constructed plane, which in turn depends upon the shape of the given general right spherical triangle. The case assumed here is that shown in Figure 70, for which (1) *angle* A *is acute and* (2) *the* *constructed perpendicular plane intersects the lines of* OA *and* OC *on the segments* OA *and* OC, *respectively.* The actual existence of this case is established by Figure 70, at least intuitively. Steps 3 and 4 of the proof below can be used to verify this with certainty. Following the derivation of the required formulas for this assumed case, all other possible cases will be exhibited. The formulas derived for the originally assumed case will then easily be shown to be valid for all cases.

Let this perpendicular plane through B meet OA in D and OC in E, where D lies between O and A, and E between O and C.

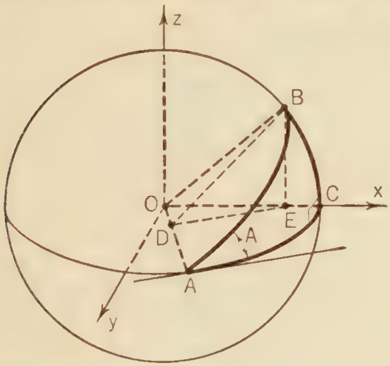

FIGURE 70

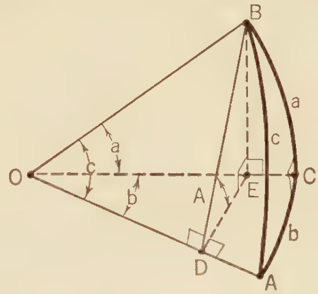

FIGURE 71

3. Plane triangles ODB and ODE (see Figure 70) are right triangles with their right angles at D:

Since OA is perpendicular to the plane BDE, by construction, the line OA is perpendicular to the two lines BD and ED in this plane and meeting OA at D. (Cf. Introduction, 3 *a*.)

4. Plane triangles OBE and DBE are right triangles with their right angles at E:

a. Since, by construction, the line OD is perpendicular to the plane BDE, the plane OAC, containing the line OD, is also perpendicular to the plane BDE. (Cf. Introduction, 4 *c*.) Reciprocally, the plane BDE is perpendicular to the plane OAC.

b. But also plane OBC is perpendicular to the plane OAC, because angle C of the spherical triangle is given a right angle.

c. Therefore, also the line BE, the intersection of the two planes BDE and OBC (each perpendicular to the plane OAC) is perpendicular to the plane OAC. (Cf. Introduction, 4 *f*.)

d. Consequently, the line BE is perpendicular to the lines OC and ED at E. (Cf. Introduction, 3 *a*.)

5. Plane angle BDE of the plane triangle BDE equals the (acute) angle A of the spherical triangle ABC:

By 3, above, the sides BD and ED of this angle are each perpendicular to the edge OA of the dihedral angle of the spherical angle A in the spherical triangle ABC, and each of these sides of the angle BDE lies in one of the faces of this dihedral angle of the spherical angle A. (Cf. Introduction, 4 *b*.)

6. Since central angles are measured by their intercepted arcs, the face angles COB, AOC, and AOB of the corresponding trihedral angle O–ABC are each equal to the side of the spherical triangle lying in the face of the respective face angle.

7. Figure 71 is Figure 70 detached from the sphere, with the facts of the above steps indicated on the figure.

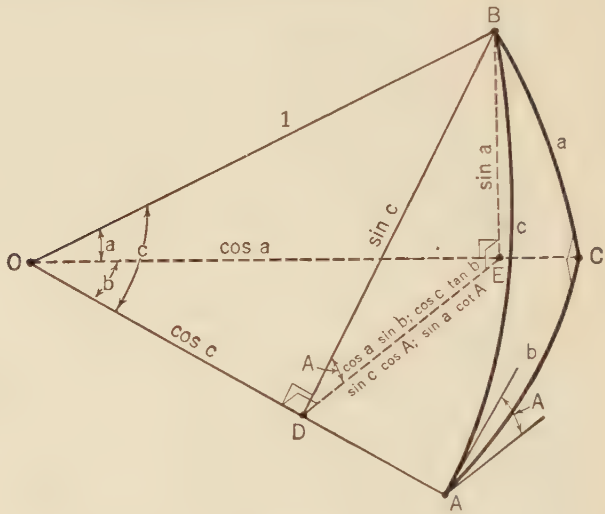

FIGURE 72

8. In Figure 72, OB is taken to be one unit long. Then, by means of the definitions of the various trigonometric functions of angles in right plane triangles (cf. Introduction, 11 *b*), the other sides of the four triangles, proved in the above to be right triangles, are shown evaluated.

Since each side of these four right triangles, OBD, OBE, ODE, BDE, is a side of two plane triangles, the evaluations of these sides, other than OB taken equal to 1, can be accomplished in several different ways. The particular plan adopted here is the following:

a. Four of these remaining five sides of the four triangles (i.e., all but DE) lie in triangles in which OB $(= 1)$ is a side. These four sides are therefore most simply evaluated by using that trigonometric function of a known angle which involves OB $(= 1)$.

b. DE, the last side of the four right triangles, can then be evaluated in four different ways — twice for each of the two right triangles of which it is a side. The two evaluations considering DE in triangle ODE are written on one side of DE, and the two evaluations considering DE in triangle BDE are written on the other side of DE.

9. Equating all possible pairs of these four expressions for DE would yield six equations involving parts a, b, c, A of the right spherical triangle ABC. Two of these equations will involve all four of these parts and, because they will therefore not lead to the expressing of *one unknown part in terms of two known parts*, are discarded. The remaining four equations, each of which contains but three of these four parts, can be most concisely written, and written so as to avoid fractions, in the following way:

$$\cos a \sin b = \cos c \tan b \quad \text{or} \quad \underline{\cos c = \cos a \cos b} \tag{1}$$
$$\sin c \cos A = \sin a \cot A \quad \text{or} \quad \underline{\sin a = \sin A \sin c} \tag{2}$$
$$\cos a \sin b = \sin a \cot A \quad \text{or} \quad \underline{\sin b = \tan a \cot A} \tag{3}$$
$$\cos c \tan b = \sin c \cos A \quad \text{or} \quad \underline{\cos A = \tan b \cot c} \tag{4}$$

10. The whole of the above procedure can be repeated by beginning with the construction of the plane through A, perpendicular to OB.* The results of such a repetition are obtained by interchanging a and b and replacing A by B in the above four formulas, giving the first formula over again and the three new ones:

$$\sin b = \sin B \sin c \tag{5}$$
$$\sin a = \tan b \cot B \tag{6}$$
$$\cos B = \tan a \cot c \tag{7}$$

11. In the seven formulas, sides a and b are each expressed in terms of two other parts of the spherical triangle in two different ways. Angles A and B and side c, on the other hand, are each expressed in terms of two other parts in but one way. We wish to rectify this discrimination and obtain two relations for each of the five parts a, b, c, A, B.

Observing that both the relations expressing a in terms of two other parts and both expressing b in terms of two other parts involve the respective parts a and b in the same function (namely, the sine), we naturally desire this duplication of functions in the second relation for parts A, B, and c. Consequently, we combine certain of the seven formulas above to obtain a second relation for each of the functions $\cos A$, $\cos B$, and $\cos c$ in terms of two other parts of the spherical triangle. One method of procedure is the following:

a. For cos A:

Multiply equations (2) and (3) together to get

$\sin a \sin b = \cos A \sin c \tan a$ or $\cos A = \cos a \sin b \csc c$.

* No restrictions (such as were assumed for A in 2 b) are logically necessary for B. The validity of the formulas so far derived could be extended to all cases now as well as later.

Eliminate sin b csc c from the above by means of formula (**5**):

$$\cos A = \cos a \sin B \tag{8}$$

b. For cos B:

An entirely similar procedure for cos B, using formulas (**5**) and (**6**) and then (**2**), yields

$$\cos B = \cos b \sin A \tag{9}$$

c. For cos c:

Multiply equations (**2**) and (**4**) together to get

$$\sin a \cos A = \sin A \cos c \tan b \quad \text{or} \quad \cos c = \sin a \cot A \cot b$$

Eliminate sin a cot b from the above by means of formula (**6**):

$$\cos c = \cot A \cot B \tag{10}$$

12. These ten formulas expressing each of the parts (aside from the right angle) of a general right spherical triangle in terms of two other parts in two different ways, enable us to solve a general right spherical triangle given any two parts besides the right angle:

$$\sin a = \sin A \sin c \tag{1}$$
$$\sin a = \tan b \cot B \tag{2}$$
$$\sin b = \sin B \sin c \tag{3}$$
$$\sin b = \tan a \cot A \tag{4}$$
$$\cos c = \cos a \cos b \tag{5}$$
$$\cos c = \cot A \cot B \tag{6}$$
$$\cos A = \cos a \sin B \tag{7}$$
$$\cos A = \tan b \cot c \tag{8}$$
$$\cos B = \cos b \sin A \tag{9}$$
$$\cos B = \tan a \cot c \tag{10}$$

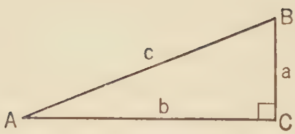

FIGURE 73

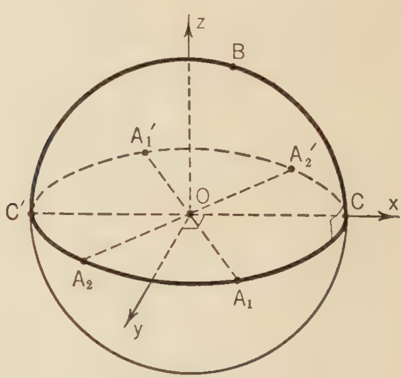

FIGURE 74

Plane right triangles are solved by an analogous set of ten formulas, derived directly from the definitions of the trigonometric functions (see Figure 73):

$$a = c \sin A, \quad b = c \sin B, \quad c = a \csc A$$
$$a = b \tan A, \quad b = a \tan B, \quad c = b \sec A$$

$$A = \sin^{-1} a/c, \quad B = \sin^{-1} b/c$$
$$A = \tan^{-1} a/b, \quad B = \tan^{-1} b/a$$

But in the case of plane right triangles the two given parts, besides the right angle, cannot be the two acute angles. Since in spherical triangles the angle sum is not constant, in contrast with plane triangles (cf. Introduction, 9 *i*), this exception does not exist in right spherical triangles.

13. It remains now to extend the validity of these ten formulas for the solution of general right spherical triangles resulting in the type of figure shown in Figure 70, and described in 2 *b*, to all general right spherical triangles.

a. Without loss of generality we can consider (see Figure 74):

(1) the right-angle vertex C, at the right-hand intersection of the horizontal great circle and that vertical great circle which is in the plane of the paper;

(2) the vertex A, somewhere on the forward half of the horizontal great circle; and

(3) the vertex B, somewhere on the upper half of that vertical great circle which is in the plane of the paper.

b. The case in which the vertex A is 90° of arc from C will be considered under special right spherical triangles. There remain but two general positions for A: less than 90° from C (A_1), and more than 90° from C (A_2). Let C', A_1', A_2' be the other extremities of the diameters through C, A_1, and A_2, respectively.

c. The restrictions on the figure for which the ten formulas have been derived were restrictions on (1) the quadrant of angle A and (2) the position of the plane from vertex B perpendicular to the diameter of vertex A. (See Figure 70.) The quadrant of angle A will be fixed by the positions of the vertices A and B. But, since the construction of the perpendicular plane from B is unique for any given positions of vertices A and B, the possible positions of B for any assumed position of vertex A will be exhausted by considering the intersections with the half-circle on which B is agreed to lie with a plane perpendicular to the diameter of vertex A as the plane moves from one end of this diameter to the other. By this procedure, for any assumed position of vertex A, all possible combinations of position of the constructed plane and quadrant of angle A will be exhausted. Applying this procedure to the two essentially different positions of A (A_1, A_2), we see from Figure 75 that *there are essentially just four figures possible for the derivation of the ten formulas for the solutions of all general right spherical triangles.* The case in which the constructed perpendicular plane intersects AOA' at O is the case in which B is 90° from C. This case, like the similar one in which A is 90° from C, will come under special right spherical triangles.

Steps 3 and 4 of the derivation of the ten formulas for the assumed first figure are useful in determining the positions of B as the intersections of its half-circle with the planes perpendicular to AOA'. As an aid in this, tangents at points A are shown to fix the directions of perpendiculars to the diameters of A.

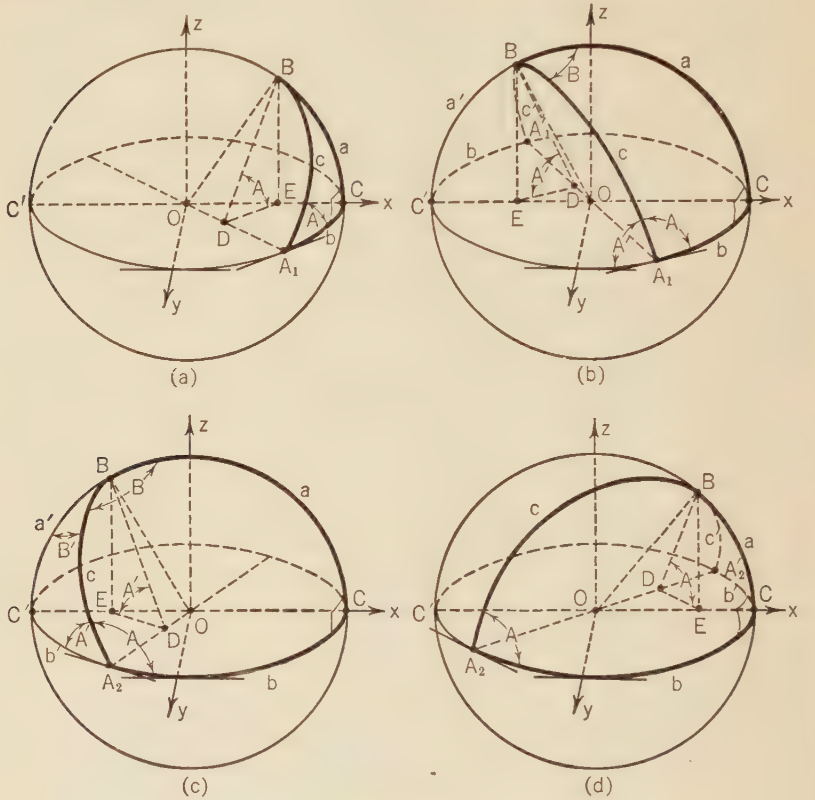

FIGURE 75

d. The case of Figure 75 *a* was the case assumed in the above derivation of the ten formulas. The extension of these formulas to the other three cases is so much the same in each case that the extension to but one case, that of Figure 75 *c*, will suffice here. Primes on letters representing spherical angles or arcs will here (as generally elsewhere in this text) indicate supplements.

e. Extension to the case of Figure 75 *c*: From the figure the general right spherical triangle A_2BC' conforms to the restrictions on that figure for which the above ten formulas have been derived. Consequently, Formulas **1-4** in 9, those which depend on the plane BDE, apply for this spherical triangle. Hence, replacing a by a', b by b', c by c, and A by A':

$$\begin{array}{lll}
\cos c = \cos a' \cos b'\,; & \cos c = (-\cos a)(-\cos b)\,; & \cos c = \cos a \cos b \\
\sin a' = \sin A' \sin c\,; & & \sin a = \sin A \sin c \\
\sin b' = \tan a' \cot A'; & \sin b = (-\tan a)(-\cot A); & \sin b = \tan a \cot A \\
\cos A' = \tan b' \cot c\,; & -\cos A = (-\tan b)(\cot c) & ;\ \cos A = \tan b \cot c
\end{array}$$

Thus these four formulas are extended. But the three new formulas derived by constructing the perpendicular plane through A can be extended in a like fashion, since the possible cases for the perpendicular plane through A are certainly precisely those for the perpendicular plane through B. Since the remaining three formulas were algebraically derived from the first seven, which are now extended, these remaining three formulas are automatically extended.

9. Napier's Rules of Circular Parts

The ten formulas given in step 12, section 8 for solving general right spherical triangles must be thoroughly committed to memory just as in the case of plane right triangles. To facilitate this memorizing, a scheme known as ***Napier's Rules*** * has been devised. Before describing this scheme the following observations about the ten formulas in question will make Napier's Rules less of a mystery.

1. The two legs are evaluated by means of the *sine* function but the hypotenuse and the adjacent angles by means of the *cosine* function.

2. The first of each pair of formulas involves either or both the *sine*, *cosine* functions on the right side but the second formula in each pair involves either or both the *tangent*, *cotangent* functions.

The scheme to memorize the ten formulas necessary for the solution of general right spherical triangles can be described as follows:

(1) Sketch and letter a general right spherical triangle as in Figure 76, indicating the right angle with a square inside the angle. Place the letters *co* before the letters for the hypotenuse (side opposite the right angle) and the angles adjacent to the hypotenuse. These parts so prefixed shall be read "co A, co c," etc., to signify that the parts A, c, etc., are to be replaced by their respective complements whenever substituted in the following rules. Cross out the letter representing the right angle.

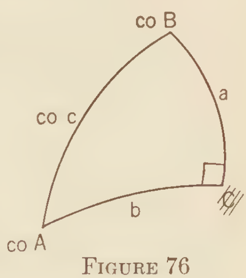

FIGURE 76

(2) Each of the remaining five parts of the right spherical triangle can in turn be considered the "middle part," with two "adjacent parts" (the parts flanking the middle part) and two "opposite parts" (the other two parts).

(3) The ten formulas can then be summarized by the rules:

The sIne of any mIddle part = product of cOsines of Opposite parts.
The sIne of any mIddle part = product of tAngents of Adjacent parts.

The repeated capitalized letters are meant to assist in committing these rules to memory by emphasizing what parts and what functions go together.

* John Napier, Laird of Merchiston (1550–1617).

Applying these rules to Figure 76 with a considered as the middle part, we have

$$\sin a = \cos \text{co } A \cos \text{co } c = \sin A \sin c \quad \text{which is formula } \mathbf{1};$$
$$\sin a = \tan b \tan \text{co } B = \tan b \cot B \quad \text{which is formula } \mathbf{2}.$$

If A is considered as the middle part we have

$$\sin \text{co } A = \cos a \cos \text{co } B \quad \text{or} \quad \cos A = \cos a \sin B \quad \text{which is formula } \mathbf{7};$$
$$\sin \text{co } A = \tan b \tan \text{co } c \quad \text{or} \quad \cos A = \tan b \cot c \quad \text{which is formula } \mathbf{8}.$$

10. Problems on Section 9

1. In the right spherical triangle ABC in which $C = 90°$, $A = 60°$, and $B = 45°$, find the hypotenuse as an arc function. Evaluate by slide rule or tables of natural functions. Sketch.

2. In the right spherical triangle ABC, C is the right angle. Find angle A as an arc function if side $b = \tan^{-1}(-3)$ and side $a = 30°$. Evaluate by slide rule or tables of natural functions. Sketch.

3. Find all sides (as arc functions) of the spherical triangle whose three angles are, respectively, $90°$, $120°$, $\cos^{-1} 0.6$. Evaluate by slide rule or tables of natural functions. Sketch.

4. Solve (i.e., find all unknown parts as arc functions) the right spherical triangle whose two legs are, respectively, $\tan^{-1} 2$, $120°$. Evaluate by slide rule or tables of natural functions.

5. Assume that the two legs of a general right spherical triangle are known.
   (a) Write the three Napier's Rules formulas each of which involves a different unknown part of the triangle and the two known parts.
   (b) If necessary, rewrite these three formulas so that each is solved for a function of one unknown part in terms of the two known parts. Underscore these three formulas in this form. They shall be called the ***working formulas*** for the solution of the particular general right spherical triangle.
   (c) Write the ***check formula*** for this particular right triangle, namely, the Napier's Rules formula involving all three unknown parts.
   (d) Substitute the three working formulas into the check formula and show that the result reduces to an identity, thus justifying the term "check formula."
   (e) Note that later, in the numerical solutions of general right spherical triangles, such check formulas will be twice applicable: (1) as a check on the working formulas before evaluation; and (2) as a check on the numerical evaluation, since the computed answers must satisfy the check formula numerically.

6. Proceed as in Problem 5 for a general right spherical triangle in which the known parts are:
   (a) The angles on the hypotenuse.
   (b) The hypotenuse and an adjacent angle.
   (c) The hypotenuse and a leg.
   (d) A leg and the angle opposite.

7. Prove that a leg of a general right spherical triangle cannot equal the hypotenuse.

8. In each of two separate figures represent a sphere by means of a circle G in the plane of the paper. In each figure let the vertex B of a right triangle ABC be at the top of G and let the hypotenuse $c = 135°$ lie along G to the left and below B. Then:

(a) In Figure 1 let the angle at B equal 30° and in Figure 2 let the angle at B equal 150°.

(b) In each figure locate a pole of the side a and, using these points, sketch the sides b of the right triangles ABC.

(c) For each figure find side b as an arc function.

(d) Comment on the answers for c in the light of the sketches.

11. Napier's Corollaries 1 and 2

All five general parts of a general right spherical triangle can be represented by an angle in either the first or second quadrant. When a particular unknown part of the triangle is to be obtained through its cosine, secant, tangent, or cotangent, there will be no uncertainty as to the quadrant of the required arc function. All four of these functions are positive in the first and negative in the second quadrant. Consequently, in this case the required part will be in the first or second quadrant according as its function is shown to be positive or negative, respectively.

When, however, a particular unknown is to be found from its sine or cosecant, we must find some way to dispel the uncertainty as to the quadrant of the required part. The sine and cosecant are positive in both first and second quadrants and, therefore, such an inverse function of a positive number can yield an angle in each quadrant. The following two relations will immediately pick out the one proper answer whenever a unique answer fails to exist on the basis of the computations alone. Since they are immediate consequences of Napier's Rules, we shall refer to them as *Napier's Corollaries.*

NAPIER'S COROLLARY 1: *In any general right spherical triangle, a leg and its opposite angle lie in the same quadrant* (*either both are less or both greater than* 90°).

NAPIER'S COROLLARY 2: *In any general right spherical triangle, the hypotenuse is in the first quadrant* (*less than* 90°) *if the quadrants of the two legs are the same* (*both less than or both greater than* 90°); *and in the second quadrant* (*greater than* 90°) *if the two legs are in different quadrants* (*one greater and the other less than* 90°).

The first corollary follows from a consideration of Formula **7**, in step 12 of section 8:

$$\cos A = \cos a \sin B, \text{ or } \sin B = \cos A \sec a.$$

Since the left side must always be positive, the signs of the factors on the right must be the same. This means that the quadrants of A and a must be the same. The like must obviously hold for B and b.

The second corollary follows from Formula **5**:

$$\cos c = \cos a \cos b.$$

If the legs are in the same quadrants, the factors in the right member will be either both positive or both negative. In either case the result will be a positive cos c and hence a first quadrant c. If the legs are in different quadrants, the signs of the factors in the right member will be opposite, yielding a negative cos c and thus a second quadrant c.

12. Napier's Corollaries 3 and 3 A*

With Napier's Rules amplified by Napier's Corollaries 1 and 2 we are in a position to solve general right spherical triangles numerically. Before proceeding to this, however, a third corollary should be discussed. It is not essential to the numerical solution of *single* right spherical triangles, but it is most helpful in the solution of general triangles by means of *pairs* of right spherical triangles. Furthermore, this third corollary provides a desirable analogue to the plane-right-triangle fact that the hypotenuse exceeds either leg.

In plane geometry, from a given point to a given line, not through the given point, there is but one perpendicular distance. This unique perpendicular distance is the shortest distance between the point and the line. In spherical geometry, however, between a given point and a given great circle, not through the point, there are, in general,† two perpendicular distances. This is so because of the fact that in spherical geometry two great circles meet in two points, whereas in plane geometry two straight lines meet in but one point. (Cf. Introduction, 6 *c*.) In Figure 77, A and a are the given point and great circle, respectively. Then, if P is the pole of a, the great circle through P and A is perpendicular to a. (Cf. Introduction, 6 *f*.) The two perpendicular distances from A to a are the supplementary arcs AD_1 and APD_2.

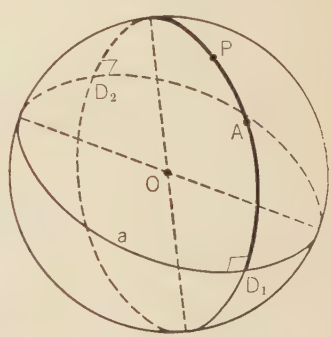

FIGURE 77

* This corollary can be postponed if desired and studied when referred to.

† When the given point is a pole of the given great circle, all great circles through the point are perpendiculars to the given great circle. (Cf. Introduction, 6 *f*.)

It is natural to inquire how the lengths of these two perpendiculars compare with the lengths of all other distances between A and a; i.e., with the lengths of all other great-circular arcs between A and a. The answer, which is the spherical-geometry analogue of the plane-geometry theorem, "The unique perpendicular is the shortest distance between point and line," is to be found in what we shall call

NAPIER'S COROLLARY 3: *Of all the great-circle distances between a general point and a general * great circle on a sphere, the two perpendicular distances are extreme distances. That is, the smaller (the one not containing a pole of the given great circle) is the least possible distance and the larger (the one containing a pole of the given great circle) is the greatest possible distance.*

Figure 78 is the same as Figure 77 with the addition of *any* other great circle through A, cutting a at C_1 and C_2. AC_1 is then the hypotenuse of two right spherical triangles, in one of which (I) the shorter perpendicular, $p = AD_1$, is a leg and in the other of which (II) the longer perpendicular, $p' = APD_2$, is a leg. By Napier's Rules for right triangle I:

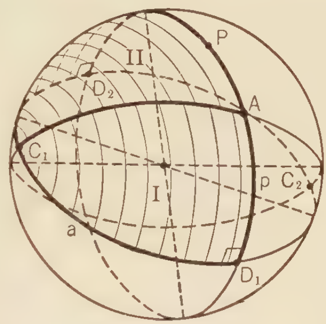

FIGURE 78

$$\cos AC_1 = \cos AD_1 \cos C_1D_1.$$

Since $\cos C_1D_1$ is numerically less than one, $\cos AC_1$ will be numerically less than $\cos AD_1$, and therefore AC_1 will be closer to 90° than is AD_1. But, since AD_1 is less than 90°, AC_1 will exceed AD_1 or, $AD_1 = p$, *the shorter perpendicular, will be less than* AC_1, which is any *other distance from* A *to* a. Similarly, in right triangle II AC_1 is closer to 90° than is the perpendicular APD_2. But, since APD_2 is greater than 90°, this means that AC_1 is less than APD_2, or $APD_2 = p'$, *the longer perpendicular, will be greater than* AC_1.

In terms of right triangles this third corollary states that:

The hypotenuse is greater than each leg if each leg is less than a quadrant.

The hypotenuse is less than each leg if each leg is greater than a quadrant.

The hypotenuse is between the two legs if one leg is less and the other leg is greater than a quadrant.

* Cf. note † on page 50. Then all great-circle distances are equal and perpendicular. (Cf. Introduction, 6 *e*.) Here, then, the two extremes coalesce, pinching all the other distances between them.

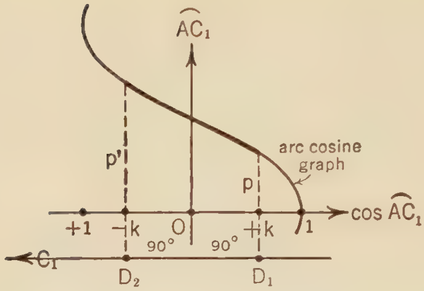

FIGURE 79

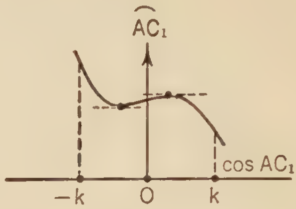

FIGURE 80

Not only are the two perpendiculars AD_1 and APD_2 extreme distances from A to $D_1C_1D_2$, but *there are no other extreme distances* from A to $D_1C_1D_2$. In other words, Figure 79 correctly shows the variation of the arc AC_1 as C_1 moves from D_1 to D_2 along the given great circle $D_1C_1D_2$. The situation in Figure 80 is impossible. Since

$$\cos AC_1 = \cos AD_1 \cos C_1D_1,$$

where AD_1 is fixed,

$$\cos AC_1 = k \cos C_1D_1,$$

where k is the constant $\cos AD_1$. This shows that AC_1 has extreme values when C_1 is at D_1 ($C_1D_1 = O$ and $AC_1 = AD_1$) and when C_1 is at D_2 ($C_1D_1 = 180°$ and $AC_1 = AD_2$) *and under no other circumstances*. The importance of this lies in our ability to state:

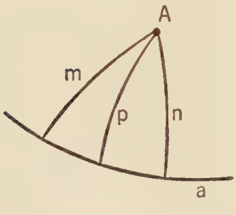

FIGURE 81

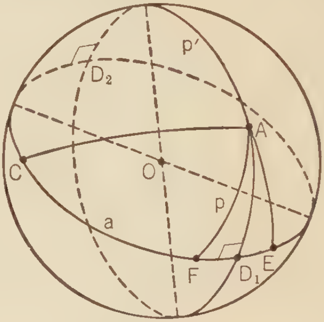

FIGURE 82

NAPIER'S COROLLARY 3 A: *If two great-circular arcs from* A *to the given great circle* $D_1C_1D_2$ *are equal, they must straddle either* p, *the shorter perpendicular from* A *to the given great circle* $D_1C_1D_2$, *or* p′, *the longer perpendicular from* A *to the given great circle* $D_1C_1D_2$, *where by*

DEFINITION: *Two arcs shall be said to* ***straddle*** *a third arc if all three arcs meet in a point, and if the third (straddled) arc lies within an angle less than* 180°, *whose sides are the first two (straddling) arcs.*

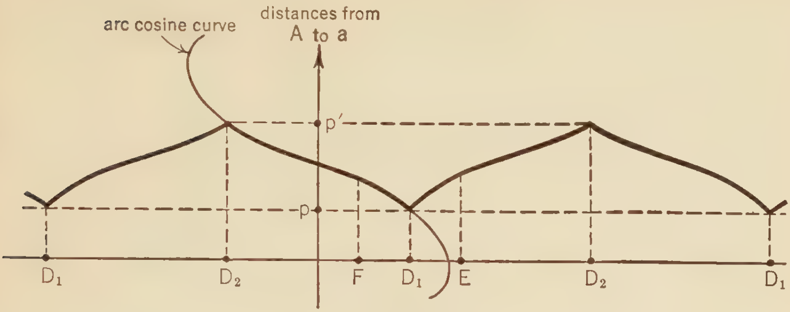

FIGURE 83

Figure 81 illustrates two arcs, m and n, straddling the arc p. Figures 82 and 83 illustrate the truth of the restatement of the third corollary. As C (on the given great circle a) moves continuously from D_1 (the foot of the shorter perpendicular from the given point A to the great circle a) to D_2 (the foot of the longer perpendicular from A to a), the arc AC varies continuously between the two and *only two* extreme values p and p' and back again to p. Then, since the distance from A to a is not constant, if arc AE equals arc AF, the arc from A to the great circle a must first decrease and then increase (or vice versa) as E moves around on the great circle a to F. In so doing these arcs must pass through an extreme value which can only be either p or p'.

13. Problems on Sections 11 and 12

1. Find the leg b of the right spherical triangle ABC in which $c = 120°$ and $B = \cot^{-1}(-\sqrt{2})$.

2. In a right spherical triangle one leg $= 150°$ and the hypotenuse $= \cot^{-1}\sqrt{2}$. Find the angle opposite the given leg explicitly and find the other unknown parts as arc functions.

3. In a right spherical triangle the hypotenuse $= \tan^{-1} 2$ and one leg $= \tan^{-1}\sqrt{3/2}$. Solve the triangle, finding explicit values for two unknowns and an arc function for the third.

4. Show that the hypotenuse of a general right spherical triangle is not necessarily the longest side.

5. Prove that the hypotenuse of an isosceles general right spherical triangle cannot exceed 90°.

DEFINITION: *An* ***isosceles spherical triangle*** *is one in which two sides are equal.* (Cf. section 20.)

Note problem 7 in section 10.

6. If the altitude from vertex A of *any* spherical triangle be drawn, state and prove by Napier's Corollaries the conditions, in terms of the angles B and C, required for this altitude to fall inside or outside the triangle.

7. By Napier's Corollaries state the conditions on the hypotenuse of a general right spherical triangle for the altitude to the hypotenuse to fall inside or outside the triangle.

8. Find the length, in terms of an arc function, of the altitude onto the hypotenuse of the right spherical triangle whose hypotenuse is 60° and one of whose angles is 135°.

9. Two great circles, G_1 and G_2, meet at an angle of 60°. A point A on G_1 is $\tan^{-1}\sqrt{2}$ from one intersection of G_1 and G_2. What is the shortest great-circle distance from A to G_2? What is the longest distance from A to G_2?

10. Two great circles, G_1 and G_2, meet at an angle of 120°. A point A on G_1 is $\cot^{-1}\sqrt{2}$ from one intersection of G_1 and G_2. What is the shortest great-circle distance from A to G_2? What is the longest distance from A to G_2?

14. Numerical Solutions of General Right Spherical Triangles

Having derived general formulas connecting parts of general right spherical triangles, we shall now show how to use these tools to compute numerical measures of unknown parts of particular general right triangles; i.e., right triangles two of whose parts other than the right angle are given numerically. Because of the length of most of the problems in spherical trigonometry and the many numbers involved, it is essential to adopt, and rigidly adhere to, certain arbitrary conventions of procedure which will be described in detail here and in later sections. The detailed nature of these conventions of procedure may at first be irksome. Experience has shown, however, that the time and pains taken to carry out these procedural details are many times repaid in the clarity of understanding of the problem and in the accuracy of numerical results.

There are three distinct parts to the procedure in solving any spherical triangle:

PART I: *Selection of proper formulas based upon a conventionalized sketch.*

PART II: *Construction of a particular* form, *based on the formulas selected, and required for the numerical computation.*

PART III: *Performance of the computation by means of logarithms, natural functions, or slide rule, as is required in the statement of the problem or as is consistent with the accuracy of the data.*

This procedure is illustrated part by part in the example below. The general description of each part of the procedure is given directly below the illustration of this particular part to emphasize the proper sequence of steps. There follow then illustrations of other examples exactly as the student will perform them; i.e., without the description of the parts of the procedure.

EXAMPLE 1: *Solve the right spherical triangle in which the hypotenuse and an adjacent angle are, respectively,* 52° 29′ 21″ *and* 172° 50′ 12″.

PART I: ***Selection of the Proper Formulas.*** See Figure 84, in which the given angle is lettered A.

$$\underline{\sin a = \sin A \sin c}$$
$$\underline{\tan b = \cos A \tan c,} \quad \cos A = \tan b \cot c$$
$$\underline{\cot B = \tan A \cos c,} \quad \cos c = \cot A \cot B$$

(1) A right spherical triangle is sketched at the right margin of the paper. This triangle is lettered, a square is placed inside the right angle C, the letter C is neatly crossed out, and the prefix "co" is written before the letters representing the hypotenuse and the adjacent angles.

(2) On this sketch the letters representing the given parts of the triangle are encircled.

(3) A Napier's Rule formula is written for each one of the unknown parts in turn. *The particular Napier's Rule formula used in each case is that one which involves the two given and the one required part.* When inspection shows (as in the case of the first formula written) that the particular required part is the *middle part*, this formula is written beginning at the left margin and is then underscored. When inspection shows (as in the case of the formulas for b and B) that the required part is *not the middle part* (i.e., the given parts do not straddle it), this formula is first written a formula's length to the right of the left margin, and it is then rewritten in the form solved for the particular unknown in question and on the same line but beginning at the left margin, where, in this solved form, it is underscored. The given parts of the triangle should be represented at this stage by their letters and not by their given numerical values.

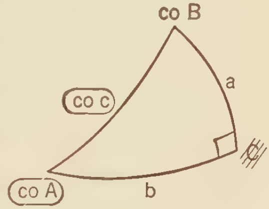

FIGURE 84

The final result of Part I of the procedure should always be, in addition to the properly labeled sketch, a set of three underscored formulas appearing on successive lines and beginning at the left margin. Each such underscored formula

(1) should be solved for one (and a different one) of the unknowns in terms of just the two given parts of the triangle;

(2) should be solved for that particular function (not its reciprocal function) appearing in the first (un-underscored) writing of the formula if the first formula had to be rewritten;

(3) should contain no fractions;

(4) should contain the known quantities on the right side of the equality sign in the same order. As a check it will be observed that no function of a known quantity is repeated.

PART II: ***Construction of the Form for Computation by Logarithms***

$A = 172° 50' 12''$ log sin   log cos (−)   log tan (−)

$c = 52° 29' 21''$ log sin______   log tan______   log cos______

$a =$
$b =$
$B =$

log sin

log tan (−)

log cot (−)

(1) The letters representing the two known parts are equated to their given numerical values on successive lines at the left margin.

(2) At the right of each known part is written the *label* for the particular function required to evaluate the *first* of the underscored equations in Part I. If inspection of the quadrant of a known part indicates that the sign of the particular function labeled is *negative*, a minus sign in parentheses is written immediately after the label. The sum of these two log functions is then indicated by writing on the third line in this column the label of the log function of the unknown being evaluated by this first underscored equation. If this sum involves an odd number of minus signs, a minus sign in parentheses is placed immediately after this label to indicate that the value of this particular unknown must be in the second quadrant. At the left margin of this third line there is then written the letter representing the unknown in the first underscored equation and an equality sign is placed after it.

(3) Leaving spaces adequate for the later writing in of the values of the labeled log functions in column two, the above procedure is repeated for each of the other two underscored equations in Part I. In order to assign a separate line for each part of the triangle, the sum of the log functions and the letter representing the corresponding unknown are written on the fourth line for the unknown in the second underscored equation and on the fifth line for the unknown in the third underscored equation.

(4) A rectangle is drawn around the labeled spaces for the unknown parts of the triangle to indicate that these are to be the answers.

Features to be noted at this stage:

1. Each part of the triangle has a line devoted to it solely.

2. Equality signs are used only in the first column to label the actual values of the parts of the triangles. In the other columns the particular log functions appearing there label the numbers to be filled in later from tables as the corresponding log functions of the part of the triangle *on the same line.* Hence, no equality signs are needed in these other columns.

3. Minus signs in parentheses are placed after all log-function labels if the corresponding trigonometric functions themselves are known to be negative.

4. No log function is yet evaluated from tables at this stage but adequate spaces are left for this purpose. *This evaluation from tables (Part III) is not to be started until the entire log form is complete.*

PART III: ***Logarithmic Evaluation***

$A = 172° 50' 12''$	log sin 9.09587	log cos (−) 9.99659	log tan (−) 9.09927
$c = 52° 29' 21''$	log sin 9.89940	log tan 10.11485	log cos 9.78455
$a = 174° 40' 35''$	log sin 8.99527		
$b = 127° 43' 42''$		log tan (−) 10.11144	
$B = 94° 22' 34''$			log cot (−) 8.88382

Features to be noted at this stage:

1. All the required log functions of the *first given* part of the triangle are to be evaluated before any log function of the second given part is looked up. In other words, the log form is to be filled in by *rows, not by columns.*

2. If an unknown is evaluated from a log function marked with a minus sign, that unknown must be in the second quadrant.

3. The quadrants of b and B are determined from the signs of their respective functions.

4. *The quadrant of* a *is not determined from its function but by means of Napier's Corollaries.* This will always be the procedure for answers evaluated from the sine or cosecant.

5. The minus tens implied after each of the logarithms in the above solution are omitted because of lack of space and because a person familiar with logarithms readily understands that these minus tens are implied.

Slide-Rule Evaluation

If the data of a right spherical triangle to be solved are given only to within multiples of ten or fifteen minutes, computation by slide rule will give answers about as accurate as the data warrant. One advantage of slide-rule technique is the rapid practice in familiarizing the student with the theory, since so much less time is taken in the actual computation than when logarithms are used. The following example illustrates the proper procedure in the case of slide-rule computation.

EXAMPLE 2: *Solve with slide rule the right spherical triangle whose hypotenuse and one leg are respectively* 78° 30′ and 113° 20′.*

Referring to Figure 85, in which the given leg is lettered b, we have

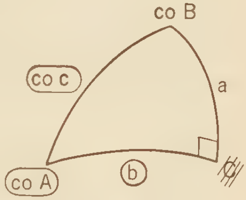

FIGURE 85

$$\underline{\cos A = \tan b \cot c}$$

$$\underline{\cos a = \sec b \cos c} \qquad \cos c = \cos a \cos b$$

$$\underline{\sin B = \sin b \csc c} \qquad \sin b = \sin c \sin B$$

$$\cos A = \tan 113^\circ 20' \cot 78^\circ 30' = -\frac{\cot 78^\circ 30'}{\cot 66^\circ 40'}; \qquad A = 118^\circ 10'.$$

$$\cos a = \sec 113^\circ 20' \cos 78^\circ 30' = -\frac{\cos 78^\circ 30'}{\cos 66^\circ 40'}; \qquad a = 120^\circ 10'.$$

$$\sin B = \sin 113^\circ 20' \csc 78^\circ 30' = \frac{\sin 66^\circ 40'}{\sin 78^\circ 30'}; \qquad B = 110^\circ 30'.$$

Features to be noted:

1. The procedure in Part I (the selection of the proper formulas) is exactly as in example 1 and is to be followed in all examples regardless of the method of computation of the numerical results.

2. Functions of angles greater than 90° were reduced to equivalent functions of angles less than 90° and all functions greater than one were replaced by reciprocals of functions less than one, as such angles and functions are more easily dealt with on the slide rule.

3. *The angle* B *was chosen in the second quadrant because of Napier's Corollary 1 and because the opposite side* b *was given in the second quadrant.*

* A discussion of this combination of data will be made in due course in section 16 a.

4. A separate line was used for the computation of each unknown and this computation was completed on this one line.

5. Answers, as always, were enclosed by a rectangle.

If the trigonometric functions of the given parts of a right spherical triangle to be solved have readily computable *exact* values, the student should, of course, use these exact values in the computation, postponing as long as possible the use of a slide rule or tables of natural functions. Example 3 is a case in point.

EXAMPLE 3: *Solve the right spherical triangle whose two legs are equal, respectively, to* 150° *and p.v.* $cot^{-1} 2$.*

Referring to Figure 86, in which the 150° side is labeled a, we have

$$\underline{\cos c = \cos a \cos b}$$

$$\underline{\cot A = \cot a \sin b} \qquad \sin b = \cot A \tan a$$

$$\underline{\cot B = \sin a \cot b} \qquad \sin a = \cot B \tan b$$

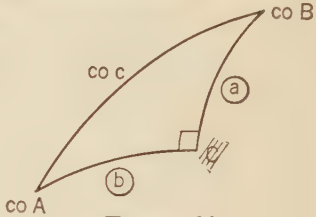

FIGURE 86

$$\cos c = \cos 150° \cos \cot^{-1} 2 = \frac{-\sqrt{3}}{2} \times \frac{2}{\sqrt{5}} = \frac{-\sqrt{15}}{5} = \begin{matrix} -0.776 = \cos c; \\ -0.7766 = \cos c; \end{matrix}$$

$$\cot A = \cot 150° \sin \cot^{-1} 2 = -\sqrt{3} \times \frac{1}{\sqrt{5}} = \frac{-\sqrt{15}}{5} = \begin{matrix} -0.776 = \cot A; \\ -0.7766 = \cot A; \end{matrix}$$

$$\cot B = \sin 150° \cot \cot^{-1} 2 = \tfrac{1}{2} \times 2 = 1 = \cot B$$

$c = 141° 00'$
$c = 140° 57'$
$A = 127° 50'$
$A = 127° 50'$
$B = 45°$

Features to be noted:

1. The selection of the proper formulas is exactly as in example 1.
2. In the actual numerical computation a separate line is used for each part to be computed, and the computation for each such part is completed on this one line.
3. The first values for c and A are obtained by slide rule and the second by tables of natural functions. Only one set need be given.
4. The answers, as always, are enclosed in a rectangle.

15. Problems on Section 14

1. In the general right spherical triangle ABC ($\measuredangle\ C = 90°$) the following parts are assumed given. Draw and properly label the figure. According to the procedure outlined above, find the formulas from which each unknown could be computed:

(a) a, B. (d) a, b.
(b) c, a. (e) c, B.
(c) A, B.

* *p.v.* ≡ "principal value." (Cf. Introduction, 17, example 1.) For positive arguments principal values are first quadrant.

2. In the general right spherical triangle ABC ($\measuredangle\, C = 90°$) the following parts are assumed given. Construct the complete form, outlined above, which should precede the actual numerical calculation of the three unknown parts:

(a) c, b. (d) c, A.
(b) A, b. (e) A, B.
(c) a, b.

3. Solve the following right spherical triangles by slide rule:

(a) $A = 70°, c = 150°$. (d) $A = 63° 20', B = 138° 30'$.
(b) $a = 50°, b = 25°$. (e) $b = 135° 40', c = 72° 30'$.
(c) $A = 110°, b = 37°$.

4. Solve the following right spherical triangles by logarithms. Check the working formulas at the outset, and later the logarithms, by means of the check formula discussed in problems 5 and 6 in section 10.

(a) $a = 67° 23' 14'', b = 18° 42' 07''$. (d) $b = 152° 00' 28'', c = 147° 13' 38''$.
(b) $A = 21° 09' 18'', c = 54° 20' 34''$. (e) $B = 93° 14' 52'', c = 175° 48' 10''$.
(c) $A = 67° 51' 15'', B = 37° 19' 37''$. (f) $c = 65° 14' 18'', A = 113° 19' 42''$.

16. Ambiguous General Right Spherical Triangles

In the illustrative examples of the preceding section the case in which the data comprised a leg and opposite angle was carefully avoided. Now consider:

EXAMPLE 4: *Solve the right spherical triangle in which a leg and opposite angle are respectively* 124° 59′ 33″ and 101° 40′ 19″.

Referring to Figure 87, in which the given leg is labeled a, we have

$\sin b = \tan a \cot A$

$\sin B = \sec a \cos A \qquad \cos A = \cos a \sin B$

$\sin c = \sin a \csc A \qquad \sin a = \sin A \sin c$

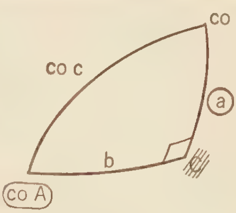

FIGURE 87

Since each required part is to be obtained from the sine function, the quadrant of each unknown part is uncertain. The data are not sufficient for invoking Napier's Corollaries, for the data already satisfy the first corollary (otherwise the triangle would be impossible), and to apply the second corollary we should have to know the quadrant of one of the unknown parts.

A consideration of the geometric requirements for the solution of such a triangle will resolve its difficulties. In this connection the following definition will be useful:

DEFINITION: *When two great circles intersect on a sphere, the portion of the surface of the sphere enclosed by half of each of the two great circles is a* ***lune***. (See Figure 88.) The ***sides of the lune*** are the halves of the two great circles bounding the lune, the ***angle of the lune*** is the angle in which the sides intersect, and the ***vertices of the lune*** are the points of intersection of the sides.

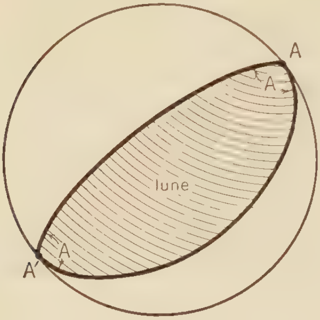

FIGURE 88

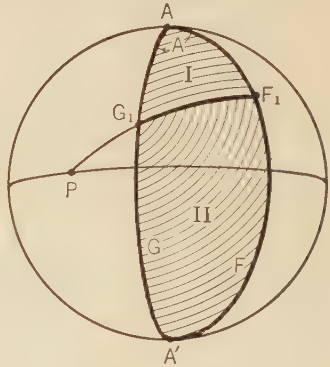

FIGURE 89

To know angle A of a general right spherical triangle to be solved is to be given a lune of angle A. In Figure 89, the sides of such a lune are represented by F and G. To be given the side a, which is to complete the right triangle, is to be required to fit this arc a between the sides of the lune so that it will be perpendicular to one of them, assumed F in Figure 89. Since (by Introduction, 6 i), all the great circles which are perpendicular to side F of the lune meet in the poles of F, the side a must lie on a great circle through P, the pole of F which is nearer side G of the lune of angle A. A question and an observation are suggested at this point:

1. Is there a great circle through P whose arc between the sides F and G of the lune of angle A is the given arc a?

2. If there be such a great circle, PG_1F_1, other than the polar of vertex A, there will be *two distinct right-triangle solutions*, AF_1G_1, $A'F_1G_1$, possessing the given parts A and a.

The answer to the above question is to be found in the following theorems, which are largely consequences of Napier's Corollary 3.

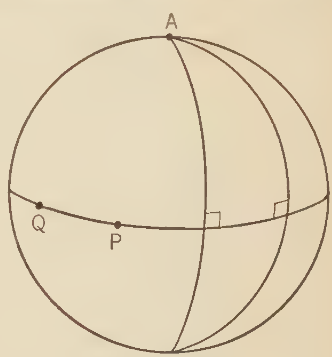

FIGURE 90

THEOREM 1: *The polar of the intersection of two great circles is the unique great circle perpendicular to both the intersecting great circles.*

This polar great circle (see Figure 90) is perpendicular to both given circles by Introduction, 6 f. Any other great circle perpendicular to both of them would have to contain their poles (by Introduction, 6 *i*) and would then be the unique great circle determined by these two poles.

THEOREM 2: *That arc of the polar of the vertex of a spherical angle which is included by the sides of the angle has an extreme value for all great-circular arcs between the sides of the angle and perpendicular to one of them. If the angle is* ***acute,*** *the included polar arc is a* ***maximum;*** *if the angle is* ***obtuse,*** *the included polar arc is a* ***minimum.***

In Figures 91 and 92, R and S are the sides of the spherical angle A. $QPGF$ is the polar of the vertex A, where P and Q are the poles of R and S, respectively, and F and G are the intersections of this polar of A with R and S, respectively.

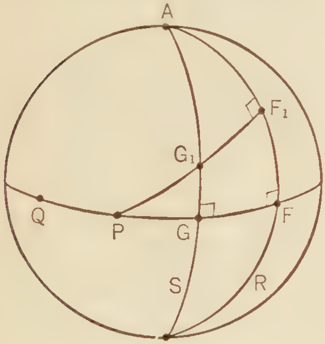

FIGURE 91

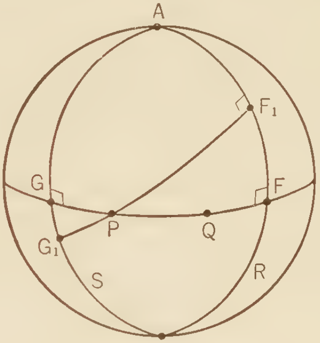

FIGURE 92

At F_1 (not at F) on R draw the arc perpendicular to R. This arc passes through P but not Q, by Introduction, 6 *i* and the above theorem. Let this arc meet S in G_1. Then:

Angle A acute (Figure 91): P is outside FG, by Introduction, 6 *j*:
$PG_1 > PG$, by Napier's Corollary 3
$PF_1 = PF$, by Introduction, 6 *e*
$G_1F_1 < GF$, by subtraction.

Angle A obtuse (Figure 92): P is inside FG, by Introduction, 6 *j*:
$PG_1 > PG$
$PF_1 = PF$
$F_1G_1 > FG$, by addition.

THEOREM 3: *A right spherical triangle for which a leg and opposite angle are given will have:*

1. ***Two*** *solutions, provided the given opposite parts are in the same quadrant, and provided the given leg is* ***less*** *than the given opposite angle when these parts are* ***acute*** *or* ***greater*** *than the given opposite angle when these parts are* ***obtuse.***

2. *One solution, a special right triangle, when the opposite parts are* ***equal.***

3. *No solution in all other cases.*

This is an immediate consequence of the previous theorem, Introduction, 6 j, and the obvious geometry of the lune.

Because right spherical triangles to be solved, given a pair of opposite parts, generally lead to two distinct solutions, this case is termed the case of ***ambiguous right triangles.*** The existence of no solution can be detected from the computations (in case the student has failed to observe this from the above geometrical discussion) by a log sine's becoming positive, or by the sine's becoming greater than one, which are obvious impossibilities. When the log sine of any computed part becomes zero, or the sine one, the case of a unique solution, a special right triangle, will be detected. In all other cases *both* the (supplementary) values, computed from the sine function, should be offered for each unknown. *These pairs of values for the computed parts must then be properly and explicitly grouped, according to Napier's Corollaries 1 and 2, to form two right-triangle solutions.*

The solution of example 4 is here completed to emphasize the need for grouping the values of the unknown sides to form actual triangles:

a = 124°59′33″	log tan (−)	10.15489	log sec (−)	10.24149	log sin	9.91340
A = 101°40′19″	log cot (−)	9.31509	log cos (−)	9.30602	log csc	10.00908
b = {17°09′51″, 162°50′10″}	log sin	9.46998				
B = {20°39′28″, 159°20′32″}			log sin	9.54751		
c = {56°46′29″, 123°13′29″}					log sin	9.92248

a_1=124°59′33″, A_1=101°40′19″; b_1= 17°09′51″; B_1= 20°39′28″; c_1=123°13′29″

a_2=124°59′33″; A_2=101°40′19″; b_2=162°50′10″; B_2=159°20′32″; c_2= 56°46′29″

The given parts are listed with both sets of the computed unknowns, in order to emphasize the application of Napier's Corollaries.

Both values of each unknown were found from the tables, which accounts for the pairs of values for the same part not always being exactly supplementary.

16a. Other General Right Spherical Triangles Leading to No Solution

The force of the term "Ambiguous" in the classification "Ambiguous General Right Spherical Triangles," those in which the data comprise a leg and opposite angle, is on the possibility of *two* solutions and not on the possibility of *no* solution. There is, in fact, one other case, not included in the term "Ambiguous General Right Spherical Triangle" and presumably already disposed of,* in which no solution may exist.

The emphasis in the so-called ambiguous case has been on the fact that the data included a pair of *opposite parts.* This should make the student suspicious of *all* such spherical triangles (a fairly safe rule), including, for instance, the case of a *right spherical triangle in which the hypotenuse is one of the given parts.* Obviously, there are but two types: (1) an angle, and (2) a leg as the second given part, in addition to the assumed right angle. Since we did not observe that these cases failed to give a figure from which Napier's Rules formulas could be derived, it would seem reasonable to apply these rules formally to the two cases under suspicion: †

To solve the right triangle given c *and* A: Referring to Figure 93:

$\sin a = \sin c \sin A$

$\cot B = \cos c \tan A, \quad \cos c = \cot A \cot B$

$\tan b = \tan c \cos A, \quad \cos A = \tan b \cot c$

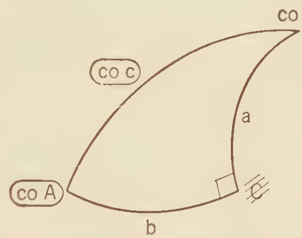

FIGURE 93

These formulas introduce no difficulties. Sin *a*, being the product of two numbers numerically not greater than one, is not greater than one, and the tangent and cotangent functions are unrestricted in value. Napier's Corollaries 1 and 2 resolve any uncertainty in the quadrant of *a*. Geometrically the exoneration of this case is immediately accomplished by noting that a unique perpendicular from *B* to the opposite side always exists.

To solve the right triangle given **c** *and* **a**: Referring to Figure 94:

$\cos B = \tan a \cot c$

$\cos b = \sec a \cos c, \quad \cos c = \cos a \cos b$

$\sin A = \sin a \csc c, \quad \sin a = \sin c \sin A$

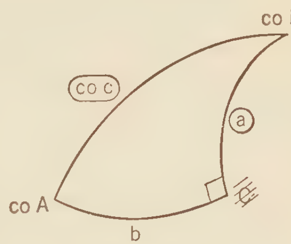

FIGURE 94

These formulas, however, certainly do not have solutions for all combinations of *a* and *c*. All unknowns are found from functions never numerically greater than 1, and each such function is the product of reciprocal functions of different arguments, which arguments could be so chosen as to make the products of the reciprocal functions numerically greater than 1. Accordingly, we see the algebraic necessity for the

THEOREM: *Right spherical triangles in which the hypotenuse and a leg are given have no solution unless the value of the hypotenuse is nearer* 90° *than is the value of the leg.*

The geometric necessity for this theorem is apparent from Napier's

* Cf. e.g. example 2, section 14.

† In any event the results deduced for these two cases can be obtained by the methods employed in Appendix I for *oblique* ambiguous triangles.

Corollary 3. The given leg is one of the two perpendiculars to the side of the other leg and, hence, must be smaller than a given acute hypotenuse or larger than a given obtuse hypotenuse. The analogous plane trigonometry theorem is obvious.

Summary of General Right Spherical Triangles:

I. *Data not containing opposite parts.*

A unique solution is always possible, as can be verified by actual geometric construction.

II. *Data containing opposite parts.*

A. *Hypotenuse part of the data.*

1. *Hypotenuse and angle given.*

A unique solution is always possible by the above.

2. *Hypotenuse and leg given.*

Unique solution, if and only if the hypotenuse is nearer than the given leg to 90°. See above.

B. *A leg and opposite angle given, "ambiguous case."*

Two, one, or no solutions according to the previous section.

17. Special Right Spherical Triangles

Reference to the ten formulas for the solution of the so-called "general" right spherical triangles (see sections 8, 12) reveals the fact that some of these formulas (those containing tangent or cotangent functions) become meaningless for certain known parts equal to 90°. It is precisely for this reason that right spherical triangles have been divided into the two classes of "general" and "special."

The ten formulas which Napier's Rules summarize are certainly not *formally* applicable to right spherical triangles containing a leg equal to 90°, a second angle equal to 90°, or both a leg and opposite angle equal to 90°. (In the first case the sine of a side is apparently infinite, in the second case a side is apparently either 0° or 180°, and in the third case the sine of a side is the product of zero and infinity — whatever this can mean!) Furthermore, in the derivation of these ten formulas precisely these cases, in which the formulas become meaningless, had to be excluded because of the degeneration of the figures on which the derivations depended, that is, the triangle BDE ceased to exist (see Figure 75). Consequently, these ten Napier's Rules formulas are *logically* as well as *formally* inapplicable.

The class of special right spherical triangles, however, can immediately be disposed of without the use of any formulas. This might well be suggested by recalling a familiar example on the earth's surface: Let the right-angle vertex C be the intersection of the equator with the northern half of a meridian. Then, if the vertex B on the equator is also a right-angle vertex, the angle A is obviously at the North Pole and the triangle is completely determined when either a or A is given as the second known part besides the assumed right angle at C.

All the above noted cases in which Napier's Rules formulas fail to apply, either formally or logically from their geometric derivation, are disposed of by the first two theorems below:

THEOREM 1: *The measure of each side [angle] of a special right spherical triangle is the same as the measure of the angle [side] opposite.*

1. Let B in Figure 95 be a second right-angle vertex in the right spherical triangle in which C is a right angle. Then the vertex A of the triangle is the pole of the side a, by Introduction, 6 *i*.

2. Then, by Introduction, 6 *e*, the theorem is proved for the parts C and c, and B and b.

3. By Introduction, 6 *j*, the theorem is true for the parts A and a.

Note that this theorem includes the case in which all the angles are right, and therefore, all six parts = 90°.

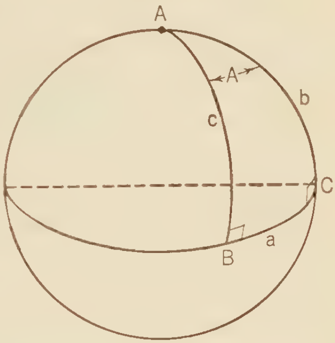

FIGURE 95

THEOREM 2: *If any of the three sides of a right spherical triangle equals* 90°, *the right spherical triangle is a special right spherical triangle. When a given* 90° *side is a leg, there is no solution possible, unless the hypotenuse also equals* 90°, *in which case there are infinitely many solutions.*

A 90° side is the hypotenuse c:

1. If the vertex A (see Figure 96 *a*) is a pole of side a, the triangle ABC is a special right spherical triangle, by definition, as angle B is then a second right angle, by Introduction, 6 *f*.

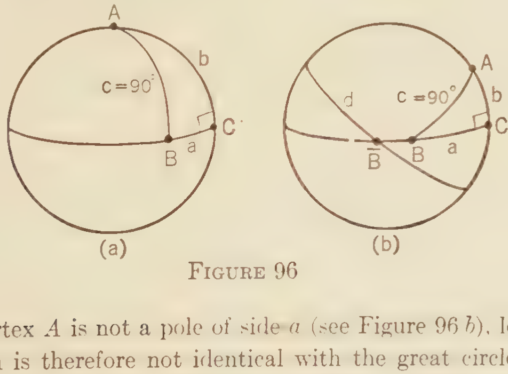

FIGURE 96

2. If vertex A is not a pole of side a (see Figure 96 *b*), let the polar of A be d, which is therefore not identical with the great circle of a. $\bar{B}$ is an intersection of d with the great circle of side a. Since $c = 90°$, B lies on d, by Introduction, 6 *e*. Therefore, B and $\bar{B}$ are identical, for each lies on d

and on the great circle of side a. But $\bar{B}$ is a pole of b, as the poles of b lie on d (by Introduction, 6 k) and on the great circle of side a (by Introduction, 6 i). Therefore, B is the pole of b and the right spherical triangle is a special right spherical triangle, by definition, as angle A is then a second right angle, by Introduction, 6 f.

A 90° side is a leg b:

1. In Figure 97 let $\bar{A}$ be the pole of side a. Then arc $\bar{A}C$ is perpendicular to side a, by Introduction, 6 f. Hence, $\bar{A}$ is on the arc AC, as there is but one great circle at C perpendicular to side a. Then, $\bar{A}$ and A are identical, as each is 90° of arc from C in the same direction on the same great circle. Therefore, A is the pole of a and angle B is a second right angle.

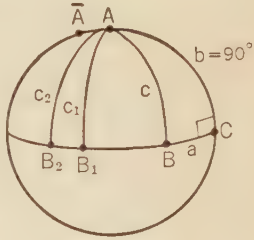

FIGURE 97

2. Since all distances from a pole to points on its polar are equal to 90°, the given hypotenuse must equal 90° for any triangle to exist. But if c is given equal to 90°, B can take on any position on the side of a.

There is one more possible case in which a pair of given parts of a right spherical triangle (besides the assumed right angle C) determines a special right spherical triangle. In this case the Napier formulas *formally* apply, and the non-existence of a figure adequate for the derivations of these formulas for this case might be difficult to detect. This case is disposed of by

THEOREM 3: *A right spherical triangle for which a pair of opposite parts have the same measure is a special right spherical triangle.* (*Converse of Theorem 1.*)

1. The case in which the opposite parts of a given same measure are the hypotenuse and the given right angle has been disposed of in Theorem 2.

2. In Figure 98 the leg a and opposite angle A are assumed of equal measure. AXA' and AYA' are two great-circular arcs meeting in the given angle A, assumed acute. The figure and reasoning for A obtuse are entirely analogous. For angle A, a right angle, the triangle is special by definition. P and Q are the poles of AXA', AYA', respectively. X and Y lie on the great circle through P and Q. Then angles at X and Y are right angles, the arc XY equals angle A, and the triangle AXY, containing the given parts angle A = side a, is a special right spherical triangle. See Introduction, 6 f, j.

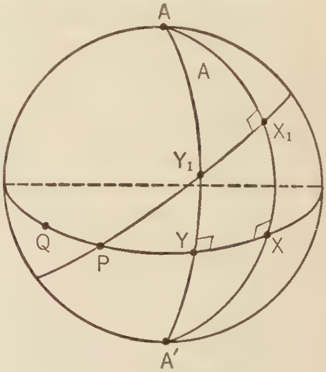

FIGURE 98

3. No other right spherical triangle containing angle A = side a is possible, for:

a. Let X_1Y_1 be any other arc between the sides of angle A and making a right angle at X_1. Then X_1Y_1 passes through P but not Q, by Introduction, 6 *i*, and hence is not perpendicular to AYA'.

b. $PY_1 > PY$ by Napier's Corollary 3.

c. By Introduction, 6 *f*, $PX_1 = PX$. Therefore $X_1Y_1 < XY$. Hence X_1Y_1 is not equal to A.

Summary of Right Spherical Triangles

Right spherical triangles have now been discussed completely. They have been divided into two classes, special and general, and the methods of solution for each class have been discussed in detail.

Given any particular right spherical triangle to solve, the student should:

1. *Classify the given triangle mentally as either special or general by means of the above three theorems, and then*

2. *Solve the given triangle by*

a. observation based on the results of the above three theorems, if the triangle is seen to be special, or by

b. computation based on Napier's Rules formulas, if the triangle is seen to be general.

If a student should immediately apply Napier's Rules to a given special right spherical triangle to be solved, he would soon be made to suspect what he should have mentally observed at the outset, namely, that the triangle, being special, should be solved mentally.

18. Problems on Sections 16 and 17

1. If a leg and opposite angle of a right spherical triangle are $\tan^{-1}(-\frac{1}{2}\sqrt{3})$, $\tan^{-1}(-\sqrt{6})$, respectively, what must the hypotenuse, other leg, and the angle opposite the other leg be to complete, with the above parts, a right spherical triangle? Express two unknowns explicitly and the third as an arc function with its quadrant specified if necessary. Sketch.

2. Find and properly group all sets of parts of the right spherical triangle ABC in which one angle and the side opposite equal 45°, 30°, respectively. Express each part either explicitly or as an arc function. In the latter case indicate the proper quadrant when the value of the particular arc function does not determine it. Sketch.

3. Using the slide rule, find all possible solutions, properly grouped, for a right spherical triangle in which a leg and opposite angle are, respectively:

(a) 25°, 28°. (c) 3° 45′, 22° 30′. (e) 127° 20′, 127° 20′.
(b) 138°, 112°. (d) 67° 30′, 48° 20′. (f) 119° 40′, 72° 30′.

4. By logarithms, find all possible solutions, properly grouped, for a right spherical triangle in which a leg and opposite angle are, respectively:

(a) 42° 15′ 25″, 53° 07′ 47″.
(b) 132° 28′ 43″, 116° 50′ 10″.
(c) 118° 17′ 25″, 93° 37′ 45″.
(d) 107° 22′ 19″, 134° 18′ 06″.
(e) 23° 00′ 42″, 78° 14′ 17″.

5. Solve the right spherical triangles in which

(a) $a = 90° 00' 00''$, $B = 132° 14' 47''$.
(b) $a = \cos^{-1}(-\frac{2}{3})$, $A = \cot^{-1}(-2/\sqrt{5})$.
(c) $b = 113° 28' 14''$, $c = 132° 14' 18''$.
(d) $c = 90° 00' 00''$, $B = 125° 18' 49''$.
(e) $a = 72° 30'$, $c = 65° 15'$.
(f) $a = 90°$, $c = 120°$.
(g) $b = \tan^{-1} 2$, $B = \sec^{-1}\sqrt{5}$.
(h) $a = 90°$, $c = 90°$.
(i) $a = \tan^{-1} 2\sqrt{2}$, $A = \cos^{-1}(-\frac{1}{3})$.
(j) $a = 90°$, $b = 90°$.
(k) $a = 138° 30'$, $c = 115° 15'$.

6. A lune of angle 42° is to be divided into two right triangles, one of which is to be twice the other in area, by a great-circle arc between the sides of the lune and perpendicular to one of them. Compute the length of this arc. (See Introduction, 9 *o*.) Sketch.

19. Quadrantal Spherical Triangles

By Introduction, 9 *f*, the polar triangle of a right spherical triangle will have a *side* = 90°.

Definition: *A spherical triangle which has a side equal to* 90° *is called a* ***quadrantal triangle.***

By the reference above, the polar triangle of a quadrantal triangle is a right triangle. Consequently, if two parts of a quadrantal triangle (other than the 90° side) are given, the triangle can be solved by solving its polar right triangle first.

Example 5: *Solve the spherical triangle in which an angle and adjacent sides are respectively* 116° 25′ 43″, 90° 00′ 00″, *and* 73° 00′ 14″. Referring to

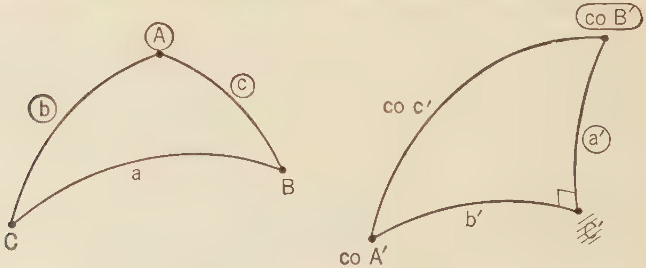

Figure 99

Figure 99, in which the labels for the given parts are A, c, and b respectively, we have

$\cos A' = \cos a' \sin B'$

$\tan b' = \sin a' \tan B'$ $\quad \sin a' = \tan b' \cot B'$

$\cot c' = \cot a' \cos B'$ $\quad \cos B' = \cot c' \tan a'$

$a' = 63°34'17''$	log cos 9.64844	log sin 9.95206	log cot 9.69638
$B' = 106°59'46''$	log sin 9.98060	log tan (−) 10.51475	log cos (−) 9.46585
$A' = 64°48'32''$	log cos 9.62904		
$b' = 108°50'48''$		log tan (−) 10.46681	
$c' = 98°16'00''$			log cot (−) 9.16223

$a =$	$115°11'28''$
$B =$	$71°09'12''$
$C =$	$81°44'00''$

Features to be noted:

1. Two figures were drawn: one for the triangle to be solved and another for its polar right triangle from which the Napier's Rules formulas were selected.
2. *Primes are used to designate parts of a polar triangle.*
3. The values of the three known parts of the polar right triangle were found by the application of Introduction, 9 *f*.

20. Isosceles Spherical Triangles

The definitions and properties of isosceles spherical triangles are exactly those for isosceles plane triangles.

Definition: ***Isosceles spherical triangles*** *are spherical triangles two of whose sides are equal. The equal sides are called the* **legs**, *the third side the* **base**, *the angles opposite the equal sides the* **base angles,** *and the angle between the legs the* **vertex angle** *of the isosceles spherical triangle.*

In general — i.e., except when the vertex is the pole of the base — there is through the vertex of an isosceles spherical triangle a unique great-circular arc perpendicular to the base and lying within the isosceles spherical triangle (see Napier's Corollary 3, *a*).

Definition: *In a general isosceles spherical triangle (one in which the vertex is not the pole of the base) the unique arc through the vertex perpendicular to the base and lying within the isosceles triangle shall be called* ***the altitude of the isosceles spherical triangle from the vertex.***

Theorem: *The base angles of an isosceles spherical triangle are equal,* *and conversely, if two angles of a spherical triangle are equal, the triangle is isosceles with the legs opposite the equal angles.*

As is shown in Figure 100, the unique altitude from the vertex of the isosceles spherical triangle divides this triangle into two right spherical triangles. The solutions of these two right spherical triangles by Napier's Rules must be identical, since in the case of either the direct or converse theorem, the two right triangles have a pair of corresponding parts respectively equal.

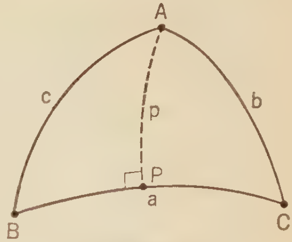

FIGURE 100

COROLLARY: *The altitude from the vertex of an isosceles spherical triangle bisects the base and the vertex angle.*

This corollary follows immediately from the proof of the above theorem.

The applications of the above theory are suggested by:

EXAMPLE 6: *Solve the spherical triangle two of whose angles are* 53° 18′ 42″ *and whose included side is* 132° 00′ 16″.

Referring to Figure 101, in which the included side is labeled c, we have:

$\cos C/2 = \sin A \cos c/2$

$\cot b = \cos A \cot c/2 \qquad \cos A = \tan c/2 \cot b.$

$A = 53° 18' 42''$	log sin 9.90412	log cos 9.77631
$c/2 = 66° 00' 08''$	log cos 9.60927	log cot 9.64853
$C/2 = 70° 57' 59''$	log cos 9.51339	
		log cot 9.42484

$b = 75° 06' 20''$
$a = 75° 06' 20''$
$C = 141° 55' 58''$

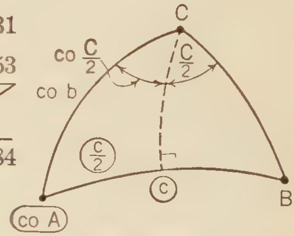

FIGURE 101

21. Problems on Sections 19 and 20

1. If two sides of a spherical triangle are 45° and the third side is 60°, find the angles as arc functions and evaluate by means of slide rule or tables of natural functions.

2. Solve the spherical triangle whose sides are respectively 45°, 90°, and 120°. Express the answers as arc functions and evaluate by slide rule or tables of natural functions.

3. Solve the spherical triangle whose sides are respectively $\tan^{-1} 2$, 90°, $\tan^{-1}(-3)$. Express the answers as arc functions and evaluate by slide rule or tables of natural functions.

4. If in a spherical triangle two angles are 135° and the third angle is 120°,

find the sides as arc functions and evaluate by slide rule or tables of natural functions.

5. Solve the equilateral spherical triangle whose sides are each 60°. Express the answers as arc functions and evaluate by slide rule or tables of natural functions.

6. If the base and vertex angle of an isosceles spherical triangle are $\cos^{-1}\frac{7}{8}$ and 120°, respectively, what could the other parts be? Express answers as properly grouped arc functions (with quadrants indicated where necessary), and then evaluate by slide rule or tables of natural functions. Cf. Introduction, 20, for the expression of the answers exactly as arc functions.

7. If a side and adjacent angles of a spherical triangle are respectively 90°, 45°, and 120°, find the other parts as arc functions and evaluate by slide rule or tables of natural functions.

8. Solve the equiangular spherical triangle whose angles are each equal to 135°. Express answers as arc functions and evaluate by slide rule or tables of natural functions.

9. Compute by slide rule or tables of natural functions that portion of the area of the whole sphere lying in the larger of the two spherical triangles for which a side and opposite angle are 135°, 150°, respectively, and a second side is 90°.

10. Solve the following spherical triangles ABC by slide rule:

(a) $a = 50°$, $b = 90°$, $c = 130°$.
(b) $a = b = c = 72°$.
(c) $a = b = c = 115°$.
(d) $A = B = C = 142°$.
(e) $a = 112°$, $B = 53°$, $C = 53°$.
(f) $a = 38°$, $b = 38°$, $A = 108°$.
(g) $a = 127°$, $b = 90°$, $A = 160°$.
(h) $a = 51° 30'$, $b = 90°$, $C = 116° 45'$.

11. Solve the following spherical triangles ABC by logarithms:

(a) $a = 71° 21' 38''$, $b = 90° 00' 00''$, $c = 154° 07' 46''$.
(b) $a = b = c = 94° 04' 44''$.
(c) $a = 55° 33' 14''$, $B = 163° 29' 43''$, $C = 163° 29' 43''$.
(d) $b = 67° 00' 40''$, $c = 90° 00' 00''$, $B = 19° 18' 00''$.
(e) $A = B = C = 62° 37' 51''$.
(f) $a = 90° 00' 00''$, $c = 110° 11' 30''$, $B = 130° 47' 20''$.

22. Order of Magnitude of Parts of a Spherical Triangle*

An interesting consequence of the above theory of isosceles spherical triangles is a very general theorem concerning *any* spherical triangle. This theorem (as was noted in the Introduction, 9 *j*) is usually proved entirely by *solid-geometry* methods, but it is more easily proved by referring to the *spherical-trigonometry* proof of the necessary properties of isosceles spherical triangles. Consequently, although this theorem is not a *necessary* part of the theory of *right* spherical triangles, it is presented here because its proof depends upon the above proved facts about isosceles spherical triangles.

THEOREM: *The order of magnitude of sides of a spherical triangle is the same as the order of magnitude of the corresponding opposite angles, or*

If A < B < C, *then* a < b < c, *and if* a < b < c, *then* A < B < C. The proof of this theorem rests upon the

* May be postponed until referred to (e.g., in section 27).

LEMMA: *If two angles of a spherical triangle are unequal, the corresponding sides opposite are unequal and unequal in the same sense, or*

If A < B, *then* a < b, *and conversely, if* a < b, *then* A < B.

1. In Figure 102, angle A is given less than angle B. At B construct angle B_1 = angle A and such that one side of angle B_1 is the side c of the triangle.

2. Then the second side of angle B_1 will cut the side b at D between C and A.

3. Then $DB = DA$ by the above theorem about isosceles spherical triangles.

Then $CD + DB = b$.

But $CD + DB > a$ by Introduction, 9 *g*.

Therefore $b > a$, or $a < b$.

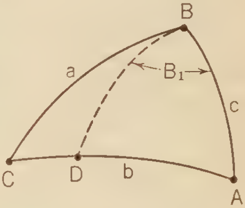

FIGURE 102

4. The converse statement is proved by the indirect method: Suppose when $a < b$ that A were equal to B. Then, by the above theorem on isosceles triangles, a would be equal to b, thus contradicting the hypothesis. Suppose when $a < b$ that A were greater than B. Then, by the above direct statement, a would be greater than b, which also contradicts the hypothesis. Consequently, the only other possibility when $a < b$, namely $A < B$, must be true.

The proof of the theorem is accomplished by applying this lemma twice—once for A and B and again for B and C.

The previously noted special case of an isosceles spherical triangle whose vertex is the pole of the base is obviously, by Introduction, 6 *f*, the case of a special right spherical triangle, for which see section 17.

23. Problems on Chapter 2

1. By Napier's Rules formulas prove the ***theorem:*** *The great circle bisecting a spherical angle is the locus of those points on the surface of the sphere which are equidistant from the sides of the angle; and the feet of the two perpendiculars from any point in this bisector of the spherical angle to the sides of the angle (one perpendicular to each side) are equally distant from the vertex of the spherical angle along the sides of the angle.* State the analogous theorem in plane geometry.

2. By Napier's Rules formulas prove the ***theorem:*** *The angle bisectors of a spherical triangle are concurrent at a point on the sphere whose great-circle distance* r *from the three sides of the spherical triangle is given by*

$$\tan \mathrm{r} = \tan \frac{\mathrm{A}}{2} \sin (\mathrm{s} - \mathrm{a}),$$

where A *is any angle of the spherical triangle,* a *is the side opposite this angle, and* s *is half the sum of the sides.* (Cf. Appendix II, section 9, for r in terms of the three sides alone.) State the analogous theorem in plane geometry.

3. By Napier's Rules formulas prove the ***theorem:*** *The great circle perpendicularly bisecting an arc of a given great circle is the locus of points on the sphere equally distant from the extremities of the given great-circular arc.* Hence, prove the ***corollary:*** *The perpendicular bisecting great-circle arcs of the sides of a spherical triangle are concurrent at a point equally distant from the vertices of the spherical*

triangle. This point of concurrence is the pole of the small circle circumscribing the plane triangle of the vertices of the spherical triangle. State the analogous theorem in plane geometry.

4. Solve the following right spherical triangles by slide rule:

(a) $b = 65° 25'$, $B = 65° 25'$.
(b) $a = 87° 42'$, $A = 91° 55'$.
(c) $b = 90° 00'$, $c = 110° 30'$.
(d) $b = 50° 10'$, $B = 73° 30'$.
(e) $a = 62° 40'$, $c = 137° 15'$.
(f) $b = 122° 30'$, $c = 72° 05'$.
(g) $a = 140° 05'$, $B = 56° 45'$.
(h) $c = 75° 50'$, $B = 102° 39'$.
(i) $b = 16° 22'$, $A = 90° 00'$.
(j) $b = 163° 35'$, $B = 132° 10'$.

5. Solve and check the following right spherical triangles by logarithms:

(a) $a = 137° 16' 37''$, $A = 137° 16' 37''$.
(b) $b = 103° 02' 44''$, $c = 80° 34' 33''$.
(c) $a = 168° 39' 51''$, $A = 114° 52' 12''$.
(d) $b = 88° 42' 08''$, $B = 98° 31' 49''$.
(e) $a = 58° 22' 29''$, $c = 141° 08' 13''$.
(f) $b = 90° 00' 00''$, $A = 76° 52' 24''$.
(g) $a = 137° 40' 28''$, $b = 100° 10' 06''$.
(h) $A = 15° 26' 31''$, $B = 88° 56' 48''$.
(i) $c = 97° 36' 14''$, $A = 123° 45' 43''$.
(j) $b = 32° 04' 57''$, $B = 44° 28' 42''$.

CHAPTER 3

The Six Types of General Triangle Solutions

24. The General Plan of Attack

DEFINITION: *Spherical triangles which are neither right, quadrantal, nor isosceles shall be termed* ***general spherical triangles.***

Any general spherical triangle can be solved if any three of its six parts are given.* If any proper combination of three parts of a triangle is given, one, or at most two, triangles are geometrically determined, and hence the remaining three parts are computable. This statement will be verified by deriving below a method for solving each one of the six possible types of spherical triangles.

The six types of solution can be accounted for in the following enumeration, in which *s* stands for a *given side* and *a* for a *given angle* of a spherical triangle:

1 Case. No sides given: Three angles given:

a.a.a.

(Figure 103)

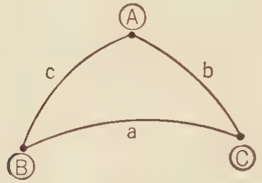

FIGURE 103

* The exception to this in *plane* trigonometry of no triangle's being determined when the three given parts are all angles is not an exception in *spherical* trigonometry. In plane trigonometry there is a definite relation between the angles of a triangle (their sum = 180°) which determines the third angle from the other two. In spherical trigonometry, on the other hand, the relation between the angles (Cf. Introduction, 9 *i*) is not precise enough to determine the third angle from the first two.

2 Cases. One side given:

- Two angles given adjacent to the given side: *a.s.a.* (Figure 104)

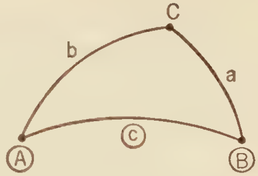

FIGURE 104

- Two angles given not adjacent to the given side: *s.a.a.* (Figure 105)

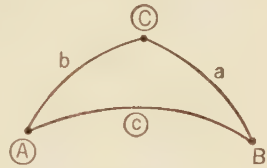

FIGURE 105

2 Cases. Two sides given:

- One angle given included by the two given sides: *s.a.s.* (Figure 106)

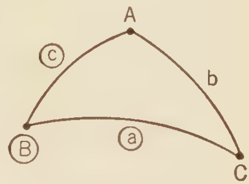

FIGURE 106

- One angle given not included by the two given sides: *a.s.s.* (Figure 107)

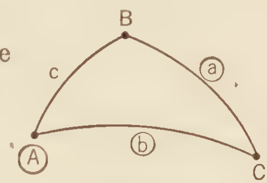

FIGURE 107

1 Case. Three sides given: No angles given: *s.s.s.* (Figure 108)

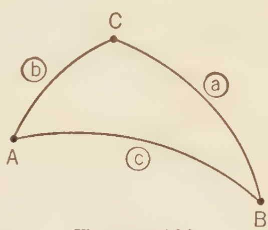

FIGURE 108

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6 Cases, total.

We shall treat these six types in three groups of two each as follows:

GROUP I: S.A.S. AND A.S.A., yielding one solution.
GROUP II: A.S.S. AND S.A.A., yielding no, one, or two solutions.
GROUP III: S.S.S. AND A.A.A., yielding one solution.

All types of general spherical triangles will be solved by the same method: ***dropping an altitude to divide the triangle into two* right *triangles, whose solutions by Napier's Rules will yield the solution of the general triangle.***

There are many other methods of solution varying for the particular cases of general spherical triangles. In fact there are several different methods for each case. The use of these other methods involves the complicated derivation of many new formulas, memorizing them, and knowing what formulas are to be used for the various types of general triangles. The question of any material lessening of numerical computation by means of these other methods is debatable. By solving *all* spherical triangles by means of the same natural construction of an altitude (if needed) and the easily remembered Napier's Rules for right triangles, the student has a simple and unified theory for the solution of any spherical triangle. Spherical trigonometry (in contrast with plane trigonometry) is too poor in theoretical ideas to warrant deriving the formulas of these alternate methods of triangle solution for their own sakes. These formulas have no applications aside from spherical triangle solutions, which can be as easily accomplished by the one essential idea: the right triangle. Many of these alternate methods of solution of general spherical triangles are derived, discussed, and applied in Appendix II. Here these alternate methods are available for the more curious student to judge for himself of their value to him after he has first mastered the unified right triangle method advocated here for the solution of all spherical triangles.

The several cases of general triangle solution are best described by showing illustrative examples worked out, followed by lists of statements calling attention to the important features of the method. The figures drawn are in many cases conventionalized diagrams in the sense that, in general, no attempt will be made to picture obtuse angles as obtuse or sides greater than a quarter circle as such. *Furthermore, altitudes will always* at first *be assumed to fall* inside *triangles. If, when part way through a problem, this is seen to be incorrect, a second figure taking this into account will be drawn.* The figure must always be drawn, but primarily to show the positions of the given parts relative to those to be computed.

25. Group I: s.a.s. and a.s.a.

Consider first the s.a.s. case. The sole purpose of the construction is to draw an altitude of the triangle so as to produce a right triangle two of whose parts are given parts of the general triangle to be solved. Obviously, the altitude from either of the two unknown angles will accomplish this. The plan of attack (cf. Figure 109) will then be:

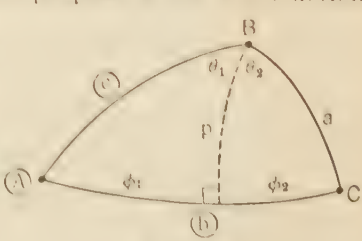


FIGURE 109

1. Find p, ϕ_1, and θ_1 in the right triangle in which A and c are given parts of the general triangle to be solved.

2. Find ϕ_2 from given b and computed ϕ_1.

3. Knowing p and ϕ_2 in the second right triangle, solve this right triangle for C, a, and θ_2.

4. Find angle B from computed θ_1 and θ_2.

In drawing the altitude to form two right triangles three possibilities come to mind: those of Figures 110, 111, and 112.

FIGURE 110

FIGURE 111

FIGURE 112

The possibility in Figure 112 will never be assumed because it can always be avoided. This is the case in which the altitude meets the given side "extended backward" and is suggested when the given angle is obtuse. But by extending the arc of the perpendicular from B in the other direction from B, Figure 110 or 111 will be obtained, as is shown in Figures 113 b, a, respectively. Since the intersections of the two perpendiculars from B to b are exactly 180° of arc apart, replacing one perpendicular by the other will not give right triangles containing sides greater than 180°.

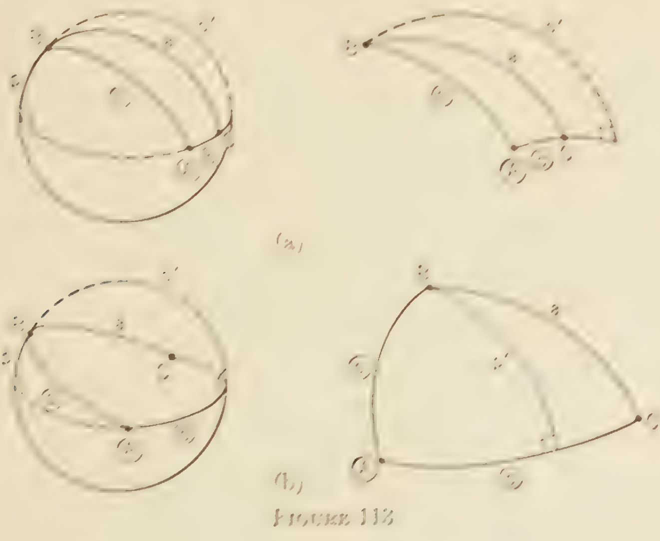


FIGURE 113

Of the two remaining possibilities (those of Figures 110 and 111) we shall always *assume* the simpler — that of Figure 110; that is, that the altitude falls *inside* the triangle. When this is not actually the case for a given triangle to be solved, the computed arc ϕ_1 will be found to be larger than b. As soon as this happens a second sketch resembling Figure 111 should be drawn which will then make clear the procedure for finding ϕ_2 from b and ϕ_1, and B from θ_1 and θ_2. *Always draw the sketch assuming the altitude inside the triangle to the left of the center of the page, and leave blank sufficient space to the right for the corrected sketch, in case this becomes necessary.*

EXAMPLE 7 (S.A.S.): *Solve the spherical triangle in which one angle is* 113° 48′ 12″ *and the two adjacent sides are, respectively,* 79° 13′ 41″ *and* 103° 07′ 23″.

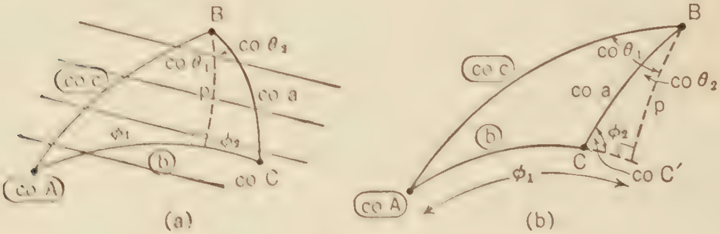


FIGURE 114

Referring to Figure 114, in which the given angle is labeled A, we have

$$\sin p = \sin A \sin c$$

$$\tan \phi_1 = \cos A \tan c \qquad \cos A = \tan \phi_1 \cot c$$

$$\cot \theta_1 = \tan A \cos c \qquad \cos c = \cot \theta_1 \cot A$$

~~$\phi_2 = b - \phi_1$~~ $\qquad \phi_2 = \phi_1 - b$

$$\cos a = \cos p \cos \phi_2$$

$$\cot \theta_2 = \sin p \cot \phi_2 \qquad \sin p = \tan \phi_2 \cot \theta_2$$

$$\cot C^{(\prime)} = \cot p \sin \phi_2 \qquad \sin \phi_2 = \tan p \cot C$$

~~$B = \theta_1 + \theta_2$~~ $\qquad B = \theta_1 - \theta_2$

$A = 113° 48' 12''$	l sin 9.96139	l cos (−) 9.60595	l tan (−) 10.35544
$c = 79° 13' 41''$	l sin 9.99228	l tan 10.72068	l cos 9.27160
$p = 115° 59' 50''$	l sin 9.95367	l tan (−) 10.32663	l cot (−) 9.62706
$\phi_1 = 115° 14' 18''$			l cos (−) 9.64180 l cot (−) 9.66812
$\theta_1 = 112° 57' 38''$			
$b = 103° 07' 23''$			
$\phi_2 = 12° 06' 55''$	l cot 10.66826		l cos 9.99022 l sin 9.32195
$\theta_2 = 13° 25' 56''$	l cot 10.62193		l cos (−) 9.63202 l cot (−) 9.01006

$a = 115° 22' 34''$
$(C' = 95° 50' 35'')$
$C = 84° 09' 25''$
$B = 99° 31' 43''$

Features to be noted:

1. The aim of the construction of the altitude p is to obtain a right triangle *in which two parts are known. Hence the altitude must not be drawn from the known angle.*

2. In the first figure, drawn to the left, the altitude was assumed to fall inside the triangle. Since, in this particular problem, this assumption was later shown to be incorrect (as ϕ_1 developed to be greater than b), the second figure was drawn in the place previously left blank to the right of the first.

3. The formulas were developed from Napier's Rules applied to the first figure since the second figure was not found to be the true one until numerical values for some of the parts had been obtained.

4. The logarithmic form, with minus signs where needed, was completed before any logarithms were looked up.

5. Here *computed* parts have to be used to find other parts. This practice is to be used as infrequently as possible.

6. The quadrant of p was determined by Napier's Corollary I.

7. The originally selected formulas for ϕ_1, B, and C were shown to be incorrect as soon as the second figure was seen to be the true one. These first formulas for ϕ_1 and B were then neatly crossed out and replaced, on the same respective lines, by the corresponding correct formulas, those based on the second figure. The original formula for C was revised by placing a prime in parentheses on C to indicate that the original formula gives not C but its supplement. ***Primes shall always indicate supplements.*** The prime was placed in parentheses to indicate that the original formula had been changed (by the addition of the prime on C) without rewriting. Accordingly C was found as the supplement of C'.

8. *Napier's Corollaries do **not** hold for non-right triangles. Angles and sides opposite in **general** triangles are not necessarily in the same quadrant.*

9. The lettering is suggestive. ϕ_1 and θ_1 are in the first right triangle used; ϕ_2 and θ_2 in the second. The ϕ's are sides, and the θ's are angles.

The procedure for the a.s.a. case is entirely similar to that for the s.a.s. case. The unique assumption as to the type of sketch to draw, with proper reservations about correcting it if need be, hold exactly as in the s.a.s. case. All that is needed is to encircle B and remove the circle from b in the above discussion. The differences will be that θ_2 will be found from B and θ_1 (instead of B from θ_1 and θ_2), and b will be found from ϕ_1 and ϕ_2 (instead of ϕ_2 from b and ϕ_1). Furthermore, the second triangle will be solved given a leg and an angle, instead of two legs.

26. Problems on Section 25

1. Solve the following spherical triangles by means of the slide rule:
   (a) $a = 53° 10'$, $B = 72° 20'$, $c = 34° 40'$.
   (b) $a = 115° 15'$, $B = 128° 45'$, $C = 73° 30'$.
   (c) $b = 87°$, $C = 25°$, $a = 62°$.
   (d) $B = 41° 00'$, $C = 113° 30'$, $a = 144° 15'$.
   (e) $a = 78° 30'$, $b = 113° 20'$, $C = 118° 10'$.
2. Solve the following spherical triangles by means of logarithms:
   (a) $a = 27° 29' 47''$, $B = 63° 43' 39''$, $c = 47° 55' 11''$.
   (b) $a = 110° 34' 50''$, $B = 61° 15' 06''$, $c = 112° 25' 14''$.
   (c) $a = 67° 53' 38''$, $B = 53° 49' 17''$, $C = 112° 33' 05''$.
   (d) $b = 117° 56' 36''$, $A = 126° 24' 42''$, $C = 52° 18' 50''$.
   (e) $c = 147° 27' 30''$, $B = 93° 45' 15''$, $a = 155° 12' 22''$.
   (f) $a = 23° 34' 50''$, $B = 117° 30' 42''$, $C = 128° 39' 13''$.
   (g) $b = 144° 31' 57''$, $A = 41° 17' 26''$, $C = 113° 00' 08''$.
   (h) $b = 142° 09' 13''$, $c = 29° 46' 08''$, $A = 137° 24' 21''$.
3. If two sides and the included angle of a spherical triangle are 23° 18′ 47″, 114° 33′ 07″, and 67° 53′ 15″, respectively, find the third side.
4. If a side and the flanking angles of a spherical triangle are 116° 17′ 25″, 23° 34′ 19″, and 70° 28′ 36″, respectively, find the third angle.
5. If two angles and the included side of a spherical triangle are respectively 78° 13′ 44″, 109° 04′ 49″, and 127° 33′ 37″, find the smaller unknown side.
6. If an angle and the flanking sides of a spherical triangle are respectively 86° 19′ 35″, 107° 34′ 13″, and 28° 56′ 15″, find the larger unknown angle.
7. Find the smaller unknown angle of the spherical triangle in which two sides and the included angle are respectively 162° 25′ 27″, 41° 04′ 19″, and 111° 18′ 47″.

27. Group II: a.s.s. and s.a.a.

The first fact to note in this group is that the triangle may be impossible because of Napier's Corollary 3 (see section 12). Figure 115 is modeled after Figure 82. C and b are respectively the given angle and the given side adjacent to the given angle. Hence, the position of the vertex A relative to the great circle on which the side a must lie is fixed. Hence, the two extreme perpendicular distances from the vertex A to the side a are fixed, and, therefore, the magnitude of the given second side c certainly must at least lie between these limits for there to be any triangle at all. Figure 115 further suggests that when *a* solution is possible there *may* be *two* solutions — one for the given second side c *on each side* of the shorter perpendicular, for instance. Since this shorter perpendicular is the least value that c could have, the length of c when on either side of p must be greater than p. Hence, it is reasonable to suppose that c could have the same given value (somewhat greater than p) for two positions, one on one side and the other on the other side of p. In each case a distinct triangle would be determined, and, hence, the solutions would involve two different triangles. If c were given exactly equal to p, there will be but one solution, a right triangle.* Appendix I describes for the more curious student exactly how the various possible kinds of solutions (double, single, or impossible) can exist in this case. Reference to this, although not essential to numerical solutions of particular triangles of this type, will make the offering of double solutions purely on the basis of numerical computation less of a blind, though numerically satisfactory, operation. As will be pointed out later in an example, for a particular numerical problem the number of solutions and their proper grouping will be ascertained automatically merely from the computations.

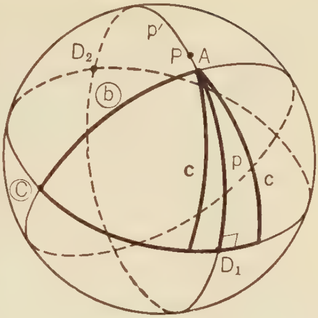


FIGURE 115

Having explicitly noted that a given a.s.s. problem to be solved is ambiguous (i.e., a case leading to no, one, or two possible solutions), the student should draw a conventionalized figure arbitrarily indicating two possible solutions. This means, by the above paragraph and Napier's Corollary 3 a, that the two assumed positions of the given side opposite the given angle should *straddle* the altitude constructed onto the unknown side (see Figure 116). With this figure as a guide the student can then mechanically perform the numerical computations for the solution of the given triangle or triangles, admitting double answers for any part whenever possible; i.e., when the part in question is evaluated from the sine or cosecant function and when Napier's Corollaries

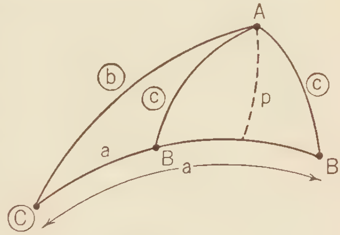


FIGURE 116

* There are, however, single solutions which are not right triangles. Some may be isosceles, but even this is not necessary. See Appendix I and example 8 *b*.

1 and 2 do not resolve the ambiguity. The figure will then show how the various sets of answers must be grouped. There will then be *no*, *one*, or *two* solutions according as the grouping of the numerical answers, dictated by the figure, gives no, one, or two *legitimate triangles*, namely, triangles (1) all of whose parts are positive and less than 180°, (2) three of whose parts are the given parts, and (3) whose parts conform to the order of magnitude theorem of section 22. The example below, which is followed by a recapitulation of the important features of the method, should make the above clear.

EXAMPLES 8 *a*, 8 *b* (A.S.S.): *Solve the spherical triangle in which an angle and an adjacent side are respectively* 122° 18′ 32″ *and* 18° 23′ 44″, *and in which the side opposite the given angle is* (a) 163° 07′ 13″; (b) 68° 20′ 23″.

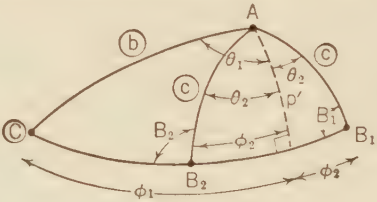


FIGURE 117

Referring to Figure 117, in which the first two given parts are labeled C and b, respectively, we have

$\sin p' = \sin b \sin C$, p obtuse, as C is.

$\tan \phi_1 = \tan b \cos C$; $\cos C = \tan \phi_1 \cot b$.

$\cot \theta_1 = \cos b \tan C$; $\cos b = \cot \theta_1 \cot C$.

$\cos \phi_2 = \sec p' \cos c$; $\cos c = \cos p' \cos \phi_2$.

$\cos \theta_2 = \tan p' \cot c$

$\sin B = \sin p' \csc c$; $\sin p' = \sin B \sin c$, *two values for* B.

$a = \phi_1 \pm \phi_2$

$A = \theta_1 \pm \theta_2$

b = 18° 23′ 44″	l sin 9.49910	l tan 9.52188	l cos 9.97722		
C = 122° 18′ 32″	l sin 9.92695	l cos (−) 9.72794	l tan (−) 10.19901		
p' = 164° 31′ 52″	l sin 9.42605			l sec (−) 10.01602	l tan (−) 9.44207
ϕ_1 = 169° 55′ 15″		l tan (−) 9.24982			
θ_1 = 146° 19′ 06″			l cot (−) 10.17623		
c = 163° 07′ 13″	l csc 10.53706			l cos (−) 9.98088	l cot (−) 10.51793
ϕ_2 = 06° 50′ 00″				l cos 9.99690	
θ_2 = 24° 13′ 00″					l cos 9.96000
B = 66° 43′ 00″, B' = 113° 17′ 00″	l sin 9.96311				

(a) (1). $b=18°23'44''$; $C=122°18'32''$; $c=163°07'13''$; $B_1=113°17'00''$; $A_1=170°32'06''$; $a_1=176°45'15''$
(2). $b=18°23'44''$; $C=122°18'32''$; $c=163°07'13''$. $B_2=66°43'00''$; $A_2=122°06'06''$; $a_2=163°05'15''$

$p' = 164°\ 31'\ 52''$ $l \sin$ 9.42605 $l \sec(-)$10.01602 $l \tan(-)$9.44207

$c = 68°\ 20'\ 23''$ $l \csc$ 10.03180 $l \cos$ 9.56715 $l \cot$ 9.59895

$B = 16°\ 40'\ 38''$, $B' = 163°\ 19'\ 22''$ $l \sin$ 9.45785

$\phi_2 = 112°\ 31'\ 06''$ $l \cos(-)$9.58317

$\theta_2 = 96°\ 18'\ 36''$ $l \cos(-)$9.04102

(b) $b=18°23'44''$; $C=122°18'32''$; $c=163°07'13''$: $B_1=163°19'22''$; $A_1=242°37'44''$; $a_1=282°26'27''$
$b=18°23'44''$; $C=122°18'32''$; $c=163°07'13''$. $B=16°40'38''$; $A=50°06'30''$; $a=57°24'06''$

Features to be noted:

1. Explicit recognition of the ambiguity of this case is made in the particular conventionalized figure drawn for it. The constructed altitude must be that onto the unknown side, in order that a right triangle may thereby be constructed having two known parts. *The ambiguity of this case is then indicated by showing two positions of the given side, opposite the given angle, straddling the constructed altitude in conformity with Napier's Corollary 3 a.*

2. The value of the constructed altitude (p or p') is always unique by Napier's Corollary 1. It is p' here as we shall let the larger of these two supplementary angles be the primed one.

3. ϕ_1, ϕ_2, θ_1, θ_2 each have unique values, as they are computed from the tangent, cotangent, and cosine functions. Combining them in different ways, purely on the basis of the figure, gives two values for a and two corresponding values for A.

4. B has two values, because it is computed from the sine function. Napier's Corollary 1 will not apply to resolve this ambiguity, since, as the figure shows, only one of the angles B of the required triangle lies in a right triangle with a leg the constructed altitude p'. But this Corollary 1 *does* show how to *group* the sets of double answers for B, a, and A: that one of the two values of B which is in the *same* quadrant as p' must be chosen for the solution triangle having its vertex on the *other side of* p' *from* C. The figure will then suggest how to group the two values for a and A with the two values of B. Note, as previously stated, that *the figure need be only schematic.* Obtuse angles need not be pictured as such, as is shown by the above use of Napier's Corollary 1 in selecting the proper one of the two possible values of B for B_1 from a consideration of the quadrants of C and the perpendicular.'

5. Recognition of the case of *no* solution would be immediate whenever log cos θ_2 is found to be positive and hence cos θ_2 greater than one. On the basis of the geometrical discussion of this ambiguous case in Appendix I, this can be predicted as soon as p or p' is computed and compared with the given c. The altitude must exceed c for C obtuse and must be less than c for C acute.

6. The case of *one* solution in example 8 *b* was recognized from the computations when one of the tentatively admitted solutions yielded a triangle

with parts greater than 180°. Single solutions are to be recognized also when a part of a proposed solution is *equal to* (as well as greater than) 180°, less than or equal to zero, or when log cos θ_2 is found to be zero (hence, cos θ_2 equal to one and θ_2 equal to zero). This latter is the case in which the single solution is a right triangle with the computed altitude equal to the given side c.

7. Note that in 8 a the law of magnitude of parts of a triangle is not sufficient to determine the grouping of the values of the computed parts to form two triangles. If the above values of B were interchanged, this law would not be violated here. The reasoning of Feature 4, however, will *always* determine the proper grouping.

The s.a.a. case, like the a.s.s. case, is ambiguous. If the s.a.a. is "polarized" (i.e., if the polar triangle of a s.a.a. case is considered), the a.s.s. case is obtained, which we know from the above to be ambiguous. Certainly, if the polar of a triangle is ambiguous, the triangle itself must likewise be ambiguous. Of course, we could then solve the s.a.a. case by polarizing into the a.s.s. case, solving this by the method just previously outlined, and then polarizing back again. Such a procedure, although perhaps simplifying the geometrical theory of oblique-triangle solutions, increases the numerical computation because of two polarizing operations. One of these is to be performed at the outset. If an error in subtraction is made here, all the ensuing logarithmic computation based on formulas derived from Napier's Rules will be void. Furthermore, understanding the actual geometrical aspects of the s.a.a. case, as distinct from the a.s.s. case, should be considered certainly as important as blindly working through a numerical solution with no regard as to just how the double solutions are possible. A complete discussion of the various figures possible in the s.a.a. case, with proofs and conditions under which they exist, is given in Appendix II for optional reference. Our present need, however, is a conventionalized sketch of double solutions in the s.a.a. case, which sketch can be used as a guide in computation. This can be arrived at intuitively by examining the sketch for the a.s.s. case with a view to building analogous relations into the s.a.a. sketch.

Figure 118 is the conventionalized sketch for the a.s.s. case. It was

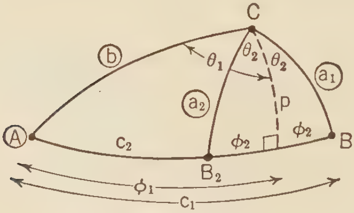


FIGURE 118

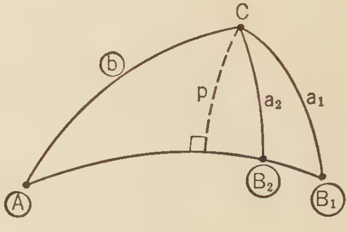


FIGURE 119

naturally suggested by the figure for this case in plane trigonometry. No such help is forthcoming in imagining the sketch for the s.a.a. case because this case in plane trigonometry is not ambiguous. The precise relation between the angles of a plane triangle (their sum = 180°) immediately determines the third angle of the triangle, thereby reducing the s.a.a. case in plane trigonometry to the a.s.a. case. But note that in Figure 118 the double solutions arise because the unknown part, B, opposite a given part, b, is computed from the formula $\sin B = \sin p \csc a$ and hence, in general, can have two different values which are supplementary. The same situation obtains in the s.a.a. case. The obvious perpendicular to be drawn to give a right triangle with two known parts is that from the vertex of the unknown angle (C in Figure 119). Then the altitude p (or p') is computed from the Napier's Rules formula $\sin p = \sin A \sin b$ and p must be taken in the same quadrant as A by Napier's Corollary 1. Then, in one of the right-hand right triangles, side a is computed from Napier's Rules formula $\sin p = \sin a \sin B$ or $\sin a = \sin p \csc B$, which shows that two solutions might arise because the unknown side a, on the basis of the computations, can have two distinct values which are supplementary.

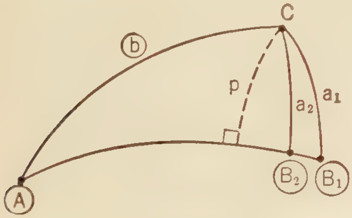


FIGURE 120

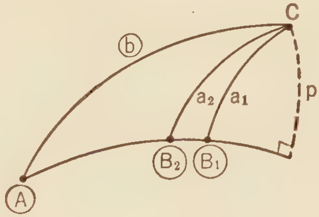


FIGURE 121

The sketch for the s.a.a. case must be emphatically distinguished from that for the a.s.s. case in one vital respect. The perpendicular in the a.s.s. case must fall inside one of the double solution triangles and outside the other, because of Napier's Corollary 3 a. The perpendicular in the s.a.a. case, by Napier's Corollary 1, must fall ***either inside both triangle solutions*** (Figure 120) ***or outside both triangle solutions*** (Figure 121). The truth of this last statement lies in the fact that otherwise (see Figure 122) the angle B would not have the same given value in the two solution triangles.

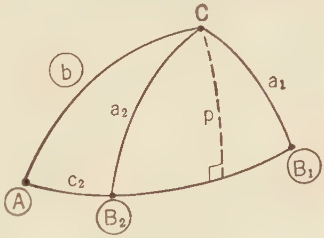


FIGURE 122

The two values for the angle at the vertex B, one for one solution triangle and the other for the other solution triangle, would then be in different quadrants, because, by Napier's Corollary 1, the angle at B in one solution triangle and the *supplement* of the angle at B in the other solution triangle would have to be in the same quadrant as the perpendicular.

Which of the two possible sketches (Figures 120 or 121) is the correct one for a particular problem is easily determined at the outset. Consequently, the proper choice should explicitly be made: *If the two given angles are in the* ***same quadrant****, then the perpendicular should be drawn* ***inside both*** *solution triangles* (Figure 120)*: if the two given angles are in* ***different quadrants,*** *the perpendicular should be drawn* ***outside both*** *solution triangles* (Figure 121). This rule *should not be memorized;* it follows immediately from Napier's Corollary 1, and, hence, should be reasoned out on each occasion.

The perpendicular must be in the same quadrant as A. Hence, if B is in the same quadrant as A, the vertices A and B in *both* triangle solutions must straddle the perpendicular; if B is in the quadrant different from that of A, the vertices A and B must be on the same side of the perpendicular in *both* triangle solutions.

Failure properly to make this obvious choice of sketch will lead to negative θ_2 and ϕ_2, from which the student might incorrectly infer that the particular problem was one for which there is no solution. If the obviously proper sketch is drawn at the outset, and if the computations are based on this sketch, then whether a particular problem has no, one, or two solutions will be determined automatically from these computations by enforcing the rule that parts of a triangle must be positive and less than 180°.

EXAMPLE 9 (S.A.A.): *Solve the spherical triangle in which a side and opposite angle are respectively* 140° 23′ 48″ *and* 139° 19′ 18″, *and in which a second angle is* 52° 10′ 20″.

Referring to Figure 123, in which the given parts are labeled b, B, and A, respectively, we have:

$\sin p = \sin b \sin A$; p in same quad. as A.

$\tan \phi_1 = \tan b \cos A$; $\cos A = \cot b \tan \phi_1$

$\cot \theta_1 = \cos b \tan A$; $\cos b = \cot A \cot \theta_1$

$\sin \phi_2 = \tan p \cot B'$; 2 supplementary values.

$\sin \theta_2 = \sec p \cos B'$; $\cos B' = \cos p \sin \theta_2$
2 supplementary values.

$\sin a = \sin p \csc B'$; $\sin p = \sin a \sin B'$
2 supplementary values.

$$\left.\begin{array}{l} c_1 = \phi_1 - \phi_2 \\ c_2 = \phi_1 - \phi_2{}' \end{array}\right\} \text{letting } \phi_2 \text{ be the acute } \phi$$

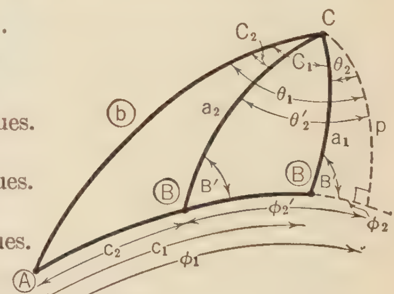


FIGURE 123

$C_1 = \theta_1 - \theta_2$, $C_2 = \theta_1 - \theta_2'$ } letting θ_2 be the acute θ

$a_1 = a$, $a_2 = a'$ } letting a be the acute a.

$A = 52° 10' 20''$	l sin 9.89755	l cos 9.78767	l tan 10.10989		
$b = 140° 23' 48''$	l sin 9.80446	l tan (−) 9.91770	l cos (−) 9.88676		
$p = 30° 13' 59''$	l sin 9.70201			l tan 9.76550	l sec 10.06349
$\phi_1 = 153° 05' 44''$		l tan (−) 9.70537			
$\theta_1 = 134° 46' 45''$			l cot (−) 9.99665		
$B = 139° 19' 18''$					
$B' = 40° 40' 42''$	l csc 10.18588			l cot 10.06576	l cos 9.87988
$\phi_2 = 42° 41' 26''$				l sin 9.83126	
$\phi_2' = 137° 18' 30''$					
$\theta_2 = 61° 22' 18''$					l sin 9.94337
$\theta_2' = 118° 37' 44''$					
$a = 50° 34' 40''$					
$a' = 129° 25' 23''$	l sin 9.88789				
$c_1 = 110° 24' 18''$					
$c_2 = 15° 47' 14''$					
$C_1 = 73° 24' 27''$					
$C_2 = 16° 09' 01''$					
$a_1 = 50° 34' 40''$					
$a_2 = 129° 25' 23''$					

$A = 52° 10' 20''$; $b = 140° 23' 48''$; $B = 139° 19' 18''$;	$c_1 = 110° 24' 18''$; $C_1 = 73° 24' 27''$; $a_1 = 50° 34' 40''$
$A = 52° 10' 20''$; $b = 140° 23' 48''$; $B = 139° 19' 18''$;	$c_2 = 15° 47' 14''$; $C_2 = 16° 09' 01''$; $a_2 = 129° 25' 23''$

Features to be noted:

1. The choice of sketch shown was deliberately made because the two given angles are in different quadrants.

2. The computations explicitly refer to this sketch:

a. B was replaced by B' (primes always indicate a supplement), because p falls outside both solution triangles.

b. ϕ_2 (first quadrant) was paired with θ_2 (first quadrant) by Napier's Corollary 1. ϕ_2' was paired with θ_2' for the same reason.

c. a_1 (first quadrant) was grouped with ϕ_2 and θ_2, by Napier's Corollary 2 (p and ϕ_2 are in the same; i.e., the first, quadrant). $a_2 = a_1'$ was grouped with ϕ_2' and θ_2' by the same corollary (p and ϕ_2' are in different quadrants).

d. c_1, c_2, C_1, and C_2 were computed as *differences* of the corresponding ϕ's and θ's, respectively. Had the perpendicular p fallen *inside* both solution triangles (B in the same quadrant as A), the c's and C's would have been computed as *sums* of the ϕ's and θ's, respectively.

3. Since both sets of answers involve only positive angles less than 180°, there are two solutions. The case of a single solution can easily be imagined: Had B been much smaller, B' might have been enough larger to make the mantissa of log sin ϕ_2 sufficiently smaller to make ϕ_2' larger than ϕ_1. Then c_2 would have been negative, or this second solution would not have existed.

Consult Appendix II for a more exact discussion of the conditions for the various kinds of solutions in this case of s.a.a.

28. Problems on Section 27

1. Solve the following spherical triangles by means of the slide rule:

(a) $C = 53°$, $b = 72°$, $c = 64°$.
(b) $A = 47° 30'$, $b = 18° 15'$, $a = 26° 45'$.
(c) $C = 115° 20'$, $b = 61° 40'$, $c = 127° 15'$.
(d) $C = 25° 30'$, $b = 113° 45'$, $c = 38° 00'$.
(e) $A = 43° 15'$, $c = 53° 15'$, $a = 136° 45'$.
(f) $B = 125° 15'$, $a = 145° 45'$, $A = 165° 45'$.

2. Solve the following spherical triangles by means of logarithms:

(a) $A = 103° 50' 19''$, $b = 162° 37' 49''$, $B = 99° 30' 53''$.
(b) $A = 53° 18' 55''$, $b = 153° 42' 08''$, $B = 136° 19' 29''$.
(c) $C = 67° 15' 15''$, $b = 72° 47' 34''$, $B = 113° 07' 04''$.
(d) $A = 86° 00' 53''$, $b = 124° 00' 34''$, $B = 39° 47' 25''$.
(e) $C = 158° 20' 34''$, $b = 168° 55' 18''$, $c = 171° 50' 50''$.
(f) $C = 86° 12' 15''$, $b = 176° 40' 10''$, $c = 10° 43' 34''$.
(g) $A = 161° 10' 52''$, $b = 167° 15' 19''$, $B = 171° 38' 38''$.
(h) $C = 99° 09' 35''$, $b = 69° 18' 47''$, $c = 110° 41' 13''$.

3. If the angles at the ends of a base of a spherical triangle are 123° 13′ 45″ and 49° 20′ 35″, respectively, and if the side opposite the smaller of these two base angles is 55° 08′ 43″, how small could the base be?

4. Find the six parts of the spherical triangle whose area is common to the two spherical triangles two of whose angles are 116° 10′ 05″, and 129° 55′ 25″, respectively, and whose side opposite the larger of these two angles is 132° 00′ 40″.

5. If two sides of a spherical triangle are to be 112° 20′ 40″ and 133° 00′ 35″, respectively, and if the angle opposite the larger of these two sides is to be 160° 50′ 15″, what proportion of the larger possible area enclosed is the smaller possible area? (Cf. Introduction, 9 *o*.)

29. Group III: s.s.s. and a.a.a.

The method of solution for this group is essentially the same as for the preceding two groups, namely, that of drawing an altitude to form two right triangles whose solutions by Napier's Rules combine to give the solution of the given triangle. However, in the cases of Group III the student should explicitly note one vital variation in this general pair-of-right-triangles solution: the altitude, because it cannot be explicitly evaluated at the outset, is to be *eliminated* from two similar Napier's Rules formulas, one applying to each of the two right triangles.* This

* The student might well ask why, in the interest of uniformity of solution of all types of general spherical triangles, this method of elimination of p was not used in Groups I and II. The answer is to be found in the greater amount of careful manipulation of Napier's Rules formulas necessary when eliminating p. Furthermore, the actual value of p in the ambiguous case of a.s.s. tells immediately whether or not the triangle is possible, and, more generally (as is shown in Appendix II), the actual value of p tells the exact nature of these ambiguous solutions.

is all the student need remember about this method, and this is suggested by the figure.

Although the s.s.s. case is perhaps the more natural of the two cases, the a.a.a. case will be described first, because its solution is a trifle simpler. The position of any desired altitude (whether inside or outside the triangle) can be immediately determined by inspection in the a.a.a.

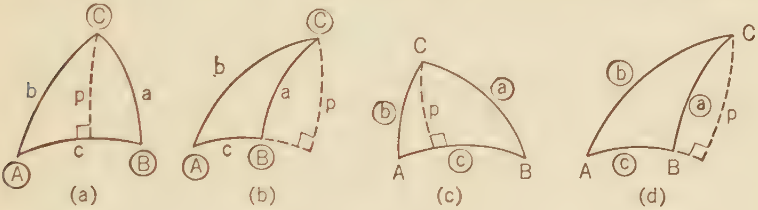


FIGURE 124

case but not in the s.s.s. By Napier's First Corollary (cf. Figure 124), it at once follows that, ***in the a.a.a. case, if the altitude* be drawn from any vertex, it will lie within or without the triangle according as the angles at the other two vertices are in the same or different quadrants.*** No such prediction can be made mentally in the s.s.s. case. Consequently, the figure in the a.a.a. case should never need revision, whereas in the s.s.s. case revision will frequently be necessary but such revisions will not affect the computations preceding the discovery of the need for this revision.

Should the student, in a given a.a.a. case, fail at the outset to notice the correct position of the constructed altitude, he could still, later on in the solution, revise his figure in this case also without nullifying his previous computations. Two changes in sign automatically cancel one another (cf. example 10). However, if the student is interested, not merely in the perfunctory results of routine computations, but also in their natural geometrical significance, he will not be satisfied to rely on this protection. He will instinctively notice the correct figure when it is immediately predictable and will then be rewarded with a satisfying correlation between the arithmetical computations and the simple and natural geometrical figure.

As in the other cases, the procedures in the a.a.a. and in the s.s.s. cases are best described by studying particular exercises for which the typical features are either described in the solution or are listed following the computations.

* The altitude from a given vertex of a triangle will, quite naturally, be taken as that one of the two perpendiculars from the given vertex to the opposite side which intersects this opposite side in a point whose distances from each of the vertices on this side are less than 180°. That is, it is the perpendicular for which both ϕ_1 and ϕ_2 are less than 180°.

EXAMPLE 10 (A.A.A.): *Solve the spherical triangle whose angles are respectively* 47° 08′ 27″, 71° 00′ 15″, *and* 108° 53′ 32″.

Electing to construct the altitude from $A = 47° 08' 27''$ (see Figure 125), we notice, by Napier's First Corollary, that it must fall outside the triangle and we accordingly draw it so. The procedure will be obvious as soon as θ_1, and therefore θ_2, is evaluated. The typical procedure to this end is: ***to find*** θ_1, ***by eliminating*** p ***from two analogous formulas for*** p, ***one for each of the two right triangles formed by the altitude, each formula involving*** p, ***a given angle, and a*** θ: Following this procedure for the definitely determined figure above:

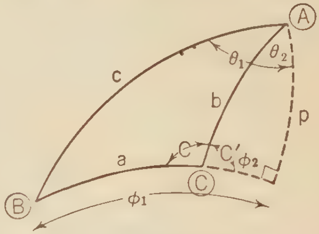


FIGURE 125

$$\left.\begin{array}{l}\cos B = \cos p \sin \theta_1 \\ \cos C' = \cos p \sin \theta_2\end{array}\right\} \frac{-\cos C}{\cos B} = \frac{\sin \theta_2}{\sin \theta_1} = \frac{\sin(\theta_1 - A)}{\sin \theta_1}$$

$$\cos C \sec B = \frac{\sin(A - \theta_1)}{\sin \theta_1} = \frac{\sin A \cos \theta_1 - \cos A \sin \theta_1}{\sin \theta_1} \; *$$

$$\cos C \sec B = \sin A \cot \theta_1 - \cos A$$

$$\cot \theta_1 = \csc A \sec B \cos C + \cot A$$

$$\theta_2 = \theta_1 - A$$

$$\cos c = \cot \theta_1 \cot B$$

$$\cos b = -\cot \theta_2 \cot C; \quad \cos b = \cot \theta_2 \cot C'$$

$$\cos \phi_1 = \cos \theta_1 \csc B; \quad \cos \theta_1 = \sin B \cos \phi_1$$

$$\cos \phi_2 = \cos \theta_2 \csc C; \quad \cos \theta_2 = \sin C' \cos \phi_2$$

$$a = \phi_1 - \phi_2$$

$A = 47°08'27''$ l csc 10.13488 n cot 0.92793

$B = 71°00'15''$ l sec 10.48745 l cot 9.53687 l csc 0.02432

$C = 108°53'32''$ l cos (−) 9.51027 l cot (−) 9.53431 l csc 0.02405

log (−) 0.13260 nat −1.35710

$\theta_1 = 113°13'36''$ n cot −0.42917 l cot(−)9.63261 l cos(−)9.59590

$\theta_2 = 66°05'09''$ l cot(−)9.64683 l cos 9.60785

l cos(−)9.16948

$c = 98°29'44''$
$b = 81°16'16''$

l cos 9.18114

$\phi_1 = 114°39'02''$ l cos(−)9.62022

$\phi_2 = 64°37'52''$ l cos 9.63190

$a = 50°01'10''$

Features to be noted:

1. Any one of the three altitudes can be drawn. The most tedious side to evaluate is that onto which the altitude is drawn. If this side should

* Cf. Introduction, 18.

not be required, values for the ϕ's would be unnecessary. Consequently, if in a particular problem a certain side is not required, let the constructed altitude be that onto this side.

2. Noticing the correct figure is a simple mental process based, not on memorizing any rule, but on an obvious application of the fundamental Napier's First Corollary. If this observation were not correctly made, the computed value of θ_1 would still be correct, for the first two lines of the formulas would then become:

$$\left.\begin{aligned}\cos B &= \cos p \sin \theta_1\\ \cos C &= \cos p \sin \theta_2\end{aligned}\right\} \quad \frac{\cos C}{\cos B} = \frac{\sin \theta_2}{\sin \theta_1} = \frac{\sin (A - \theta_1)}{\sin \theta_1}$$

which leads to the above derived formula for cot θ_1. The figure could then be revised when the computed value of θ_1 is shown to exceed the given value of A, just as in the following example in the s.s.s. case. However, it should be disturbing to have the negative ratio of cos C to cos B (since C and B are in different quadrants) set equal to the ratio of two sines which must therefore be positive for the assumed positive θ_1.

3. The formula for cot θ_1 is derived anew *for each problem.* The derivation is short, and it obviates memorizing a formula much too special to warrant memorizing. The fundamental and straightforward method alone should be borne in mind: ***Using first one and then the other right triangle, write two similar formulas, each expressing the altitude*** (p *or* p′) ***in terms of a given angle of the triangle and a θ-angle formed by a side of the triangle and the constructed altitude. Then eliminate from these two similar formulas the function involving the perpendicular and solve for a function of just one of the θ-angles in terms of functions of the three given angles.***

EXAMPLE 11 (s.s.s.): *Find the magnitude of the two largest angles of the spherical triangle in which the three sides are respectively* 53° 18′ 37″, 93° 07′ 19″, *and* 127° 00′ 43″.

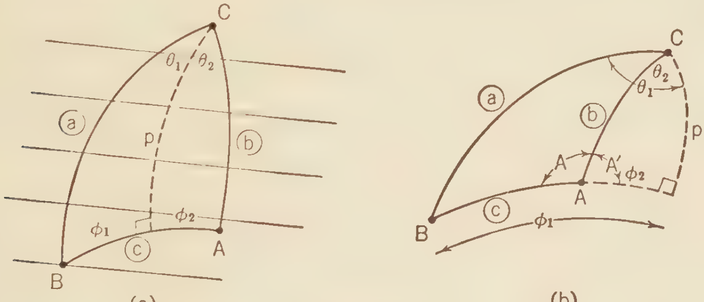


FIGURE 126

By the order of magnitude relation for spherical triangles (section 22), the required angles are those opposite the two largest sides. Since the smallest angle is not required, we shall draw the altitude from this vertex. Just as the side onto which the altitude was drawn in the a.a.a. case was the most tedious to compute, so, for similar reasons, the angle from which the altitude is drawn in the s.s.s. case is the most difficult angle to compute. The procedure in this s.s.s. case is entirely analogous to that for the previously illustrated a.a.a. case. In the present instance we must first find the ϕ's and then the θ's (the opposite order held in the a.a.a. case), because, since the sum of the ϕ's is a given part, we can immediately express one in terms of the other. However, for this particular example with the most convenient altitude drawn, the θ's need never be computed and, therefore, the obvious formulas for them in this example will not be exhibited.

As usual, when the position of the constructed altitude is not definitely predictable at the outset, we assume that the altitude lies *within* the triangle.

Referring to Figure 126, in which the given parts are labeled c, b, a, respectively, we have

$$\left.\begin{aligned}\cos a &= \cos p \cos \phi_1\\ \cos b &= \cos p \cos \phi_2\end{aligned}\right\} \frac{\cos b}{\cos a} = \frac{\cos \phi_2}{\cos \phi_1} = \begin{cases}\dfrac{\cos (c - \phi_1)}{\cos \phi_1}, & \phi_1 < c\\[2ex] \dfrac{\cos (\phi_1 - c)}{\cos \phi_1}, & \phi_1 > c\end{cases}$$

Note that, *if* the above *assumed* figure is not the actual one — i.e., if ϕ_1 is shown to be greater than the given c — then ϕ_2 is to be replaced by $(\phi_1 - c)$ instead of by $(c - \phi_1)$. But, since ϕ_2 enters only in the cosine, which is unchanged by a change in sign in the angle, the value of ϕ_1 computed from the *assumed* figure will be correct even when this assumed figure is shown not to be the actual one.

$$\cos b \sec a = \frac{\cos c \cos \phi_1 + \sin c \sin \phi_1^{*}}{\cos \phi_1}$$

$$\cos b \sec a = \cos c + \sin c \tan \phi_1$$

$\tan \phi_1 = \sec a \cos b \csc c - \cot c$

~~$\phi_2 = c - \phi_1$~~; $\phi_2 = \phi_1 - c$

$\cos B = \tan \phi_1 \cot a$

~~$\cos A = \tan \phi_2 \cot b$~~; $\cos A' = \tan \phi_2 \cot b$ *and* $A = 180° - A'$

a = 127° 00′ 43″	l sec (−) 10.22042		l cot (−) 9.87730	
b = 93° 07′ 19″	l cos (−) 8.73608			l cot (−) 8.73673
c = 53° 18′ 37″	l csc 10.09589	− n cot 0.74510		
	log 9.05239	nat 0.11282		
ϕ_1 = 147° 41′ 44″		n tan − 0.63228	l tan (−) 9.80092	
ϕ_2 = 94° 23′ 07″				l tan (−) 11.11528
B = 61° 31′ 54″			l cos 9.67822	
A' = 44° 39′ 53″				l cos 9.85201
A = 135° 20′ 07″				

Features to be noted:

1. Only such unknowns as are required are computed, and the altitude constructed is that one for which the computation of the required parts will be simplest.

* Cf. Introduction, 18.

2. The conventionally assumed figure (altitude within the triangle) is found to be incorrect when the computed value of ϕ_1 is found to exceed the given value of c. The assumed figure is then neatly crossed out and replaced on the right by the actual figure. Furthermore, those formulas in the subsequent solution which are affected by this change are replaced by the actual formulas. Were the entire solution required, the following additions to the above list of formulas would have been made with their subsequent revisions:

$\sin\theta_1 = \sin\phi_1 \csc a$; θ_1 in the same quad. as ϕ_1; $\sin\phi_1 = \sin\theta_1 \sin a$
$\sin\theta_2 = \sin\phi_2 \csc b$: θ_2 in the same quad. as ϕ_2; $\sin\phi_2 = \sin\theta_2 \sin b$
~~$C = \theta_1 + \theta_2$~~; $C = \theta_1 - \theta_2$.

30. Problems on Section 29

1. Solve the following spherical triangles by means of the slide rule:
   (a) $a = 52°$, $b = 117°$, $c = 129°$.
   (b) $A = 21° 30'$, $B = 37° 20'$, $C = 133° 45'$.
   (c) $a = 49° 15'$, $b = 67° 45'$, $c = 76°, 30'$.
   (d) $a = 13° 20'$, $b = 22° 30'$, $c = 27° 45'$.

2. Solve the following spherical triangles by means of logarithms:
   (a) $a = 57° 10' 49''$, $b = 115° 23' 08''$, $c = 126° 40' 56''$.
   (b) $A = 163° 36' 19''$, $B = 168° 17' 27''$, $C = 171° 01' 41''$.
   (c) $A = 32° 00' 40''$, $B = 93° 52' 45''$, $C = 107° 21' 25''$.
   (d) $a = 87° 08' 35''$, $b = 103° 41' 49''$, $c = 151° 09' 51''$.
   (e) $a = 41° 20' 42''$, $b = 110° 00' 52''$, $c = 123° 53' 09''$.
   (f) $A = 59° 10' 23''$, $B = 65° 18' 37''$, $C = 79° 00' 51''$.

3. Find the smallest angle of the spherical triangle in which the three sides are 57° 30′, 119° 20′, and 141° 15′, respectively.

4. Find the largest side of the spherical triangle in which the three angles are 17° 25′ 15″, 38° 40′ 21″, 144° 00′ 05″, respectively.

5. Find the altitudes from the smallest angle of the triangle in which $A = 42° 59' 07''$, $B = 67° 33' 25''$, $C = 122° 14' 35''$.

6. Find the altitudes onto the longest side of the spherical triangle in which $a = 54° 20' 55''$, $b = 107° 39' 47''$, $c = 133° 28' 19''$.

31. Problems on Chapter 3

1. Solve the following spherical triangles by means of the slide rule:
   (a) $C = 66°$, $b = 152°$, $c = 66°$.
   (b) $a = 113° 20'$, $B = 56° 30'$, $c = 142° 30'$.
   (c) $a = 47° 20'$, $b = 66° 20'$, $c = 73° 00'$.
   (d) $C = 130°$, $b = 108°$, $c = 65°$.
   (e) $A = 39° 30'$, $B = 64° 30'$, $C = 148° 00'$.
   (f) $a = 52° 20'$, $B = 138° 30'$, $C = 76° 00'$.
   (g) $A = 55° 00'$, $B = 138° 30'$, $C = 55° 00'$.
   (h) $A = 64° 30'$, $b = 141° 30'$, $C = 148° 30'$.
   (i) $a = 115° 30'$, $b = 115° 30'$, $c = 115° 30'$.
   (j) $A = 59° 45'$, $b = 106° 20'$, $B = 122° 15'$.

2. Solve the following spherical triangles by means of logarithms:

(a) $a = 23° 14' 39''$, $b = 78° 00' 47''$, $c = 82° 56' 04''$.
(b) $C = 98° 15' 40''$, $b = 81° 45' 41''$, $c = 110° 06' 07''$.
(c) $A = 163° 22' 35''$, $b = 14° 36' 51''$, $a = 169° 08' 31''$.
(d) $a = 47° 15' 44''$, $B = 134° 47' 41''$, $c = 126° 00' 35''$.
(e) $b = 112° 31' 51''$, $C = 130° 27' 50''$, $a = 155° 18' 42''$.
(f) $a = 130° 50' 10''$, $B = 152° 20' 54''$, $C = 114° 33' 27''$.
(g) $a = 172° 18' 44'$, $b = 90° 00' 00''$, $C = 110° 48' 00''$.
(h) $C = 107° 49' 23''$, $b = 74° 50' 23''$, $c = 93° 00' 47''$.
(i) $A = 73° 18' 25''$, $b = 11° 30' 45''$, $B = 171° 00' 35''$.
(j) $A = 140° 40' 50''$, $b = 33° 25' 15''$, $B = 56° 34' 45''$.
(k) $A = 86° 10' 20''$, $B = 113° 48' 24''$, $C = 141° 18' 19''$.
(l) $a = 77° 05' 45''$, $B = 18° 23' 32''$, $C = 64° 38' 54''$.
(m) $C = 47° 15' 51''$, $b = 18° 25' 41''$, $c = 18° 25' 41''$.
(n) $A = 49° 43' 19''$, $b = 49° 27' 47''$, $B = 39° 25' 25''$.
(o) $A = 107° 00' 58''$, $b = 24° 04' 50''$, $B = 18° 14' 19''$.
(p) $a = 09° 32' 50''$, $b = 09° 32' 50''$, $c = 09° 32' 50''$.
(q) $a = 114° 38' 46''$, $b = 114° 38' 46''$, $c = 114° 38' 46''$.
(r) $A = 61° 10' 24''$, $B = 61° 10' 24''$, $C = 61° 10' 24''$.
(s) $A = 58° 19' 34''$, $b = 117° 22' 20''$, $B = 117° 22' 20''$.
(t) $A = 125° 25' 25''$, $b = 145° 45' 34''$, $B = 165° 24' 47''$.
(u) $A = 54° 54' 42''$, $c = 69° 25' 11''$, $a = 50° 00' 00''$.
(v) $b = 22° 15' 07''$, $c = 55° 09' 32''$, $A = 73° 27' 11''$.

3. If an angle and the adjacent sides of a spherical triangle are, respectively, $115° 30' 27''$, $41° 21' 39''$, $145° 45' 38''$, find the shorter altitude of the triangle from the known vertex.

4. Find the largest angle of the spherical triangle in which $a = 107° 18'$, $b = 123° 25'$, $c = 53° 50'$.

5. Find the two shorter legs of the spherical triangle in which the three angles are $63° 20' 40''$, $111° 35' 25''$, $34° 35' 15''$, respectively.

6. Find the altitudes onto the shortest side of the spherical triangle whose sides are $33° 18' 49''$, $97° 34' 15''$, $111° 52' 08''$, respectively.

7. If the base of a spherical triangle is $147° 18' 41''$ and if the base angles are $82° 31' 14''$ and $53° 46' 07''$, respectively, find the shorter altitude onto the given side.

8. Find the third angle of a spherical triangle in which the other two angles are $47° 23' 41''$, $105° 48' 55''$, respectively, and the included side is $63° 18' 46''$.

Terrestrial Applications

32. Positions of Points on the Earth's Surface

The two most important spheres to which spherical trigonometry is to apply are the earth and the concentric spherical shell about the earth, called the heavens, in which we imagine the stars as fixed. The first of these spheres, the earth, will be treated in this chapter, and the second sphere, the heavens, in the next.

The earth is not exactly a sphere but sufficiently so to be considered as such. According to Bowditch,* the earth's longer or equatorial diameter measures about 7927 statute miles, and its shorter or polar diameter measures about 7900 statute miles. This means that by taking the ***radius of the earth* = 3957 *statute miles*** (approximately the mean radius), we shall never be in error by more than 0.2 per cent. Spherical trigonometry on the earth is useful when the distances are sufficiently large to make the curvature of the earth significant. This is the case in shipping on the oceans and in flying.

The familiar co-ordinate system of meridians and parallels of latitude on the earth's surface serves to locate vertices of spherical triangles and measure their angles and sides. Figure 127 will help to recall this system.

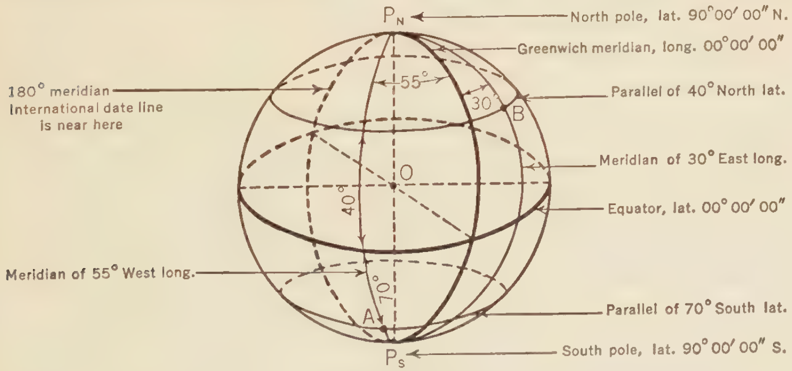


FIGURE 127

* *American Practical Navigator*, originally by Nathaniel Bowditch. Washington, D.C.: United States Hydrographic Office.

The great circle called the equator, from which parallels of latitude are measured, is the great circle whose poles, called the North Pole and the South Pole, are the ends of the axis of the earth's rotation. The great circle called the Greenwich Meridian, from which meridians of longitude are measured, is an arbitrary great circle through the poles. Its position, possessing no such geographical significance as the equator, has been adopted by international agreement as the meridian passing through Greenwich, England. The position of point A is described as lat. 70° South; long. 55° West. The co-ordinates lat. 40° N., long. 30° E. determine the point B. Any point on the earth's surface (except the north and south poles) possesses a unique pair of directed angles which exactly describes the position of the point on the earth's surface. The first angle, labeled "lat." and described as either north or south (N. or S.), must be between 0° and 90° and indicates the parallel of latitude on which the point lies or the directed number of degrees of arc, measured along the meridian of the point, separating the point from the equator. The second angle, labeled "long." and described as either east or west (E. or W.), must be between 0° and 180° and indicates the meridian of the point or the directed number of degrees between the Greenwich Meridian and the great circle through the poles and the given point. Conversely, any pair of angles, within the above described ranges, directed, and given in the order described above, will determine a unique point on the earth's surface. The north and south pole are each described by but one angle: lat. 90° N., lat. 90° S., respectively.

An essential feature of the co-ordinate system of Latitude and Longitude on the earth's surface is that, whereas the meridians are all great circles, the parallels of latitude, with the exception of the equator, are *not* great circles. Consequently, arcs of parallels of latitude (except the equator) cannot be used as sides of spherical triangles. This restriction is no hardship, as distances along parallels of latitude (except the equator), being distances along small circles, will not be minimum distances. The whole point in using spherical triangles on the earth's surface is to compute *shortest* distances between points on the earth and to describe how to traverse them. If in a particular problem the length of arc (AB in Figure 128, for instance) of some parallel of latitude should be required, it can be found

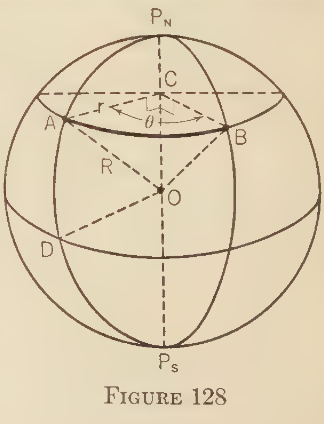


FIGURE 128

by the formula s $(= AB) = r\theta$, where r is the radius of the particular small circle of the parallel of latitude and θ is the difference (in radians) of longitude of the two points A and B. The radius, r, of the small circle of A and B can be computed from the right triangle OAC when the latitude (angle AOD = angle CAO) of the small circle and the radius, R, of the earth are known.

33. Spherical Triangles on the Earth's Surface

If any two points on the earth's surface are given, the spherical triangle most useful in problems involving these points is the spherical triangle whose vertices are the two given points and one of the two terrestrial poles. The meridian arcs between the pole and the given two

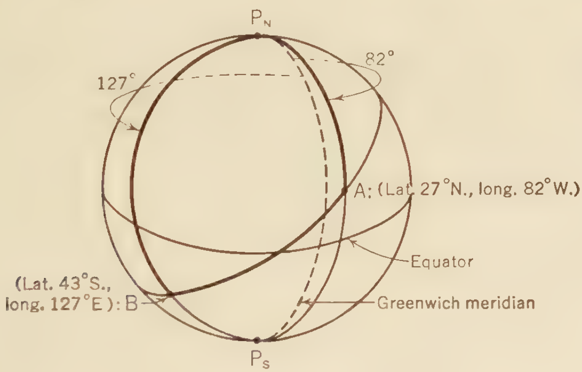


FIGURE 129

points form two of the sides of the spherical triangle, the third side being the minor arc of the great circle determined by the given two points, as is shown in Figure 129. The lengths of the sides on the meridians will therefore be either 90° minus or 90° plus the latitude of the points as the figure directs. The angle of the triangle at the pole can easily be computed from the longitudes of the given points. Since the exact form of this computation depends upon the position of the Greenwich Meridian relative to the meridians of the given points, no set formula can be derived for all cases. In any case the computation is simple and should be based on the figure for each particular case. In Figure 129 one of the more complicated cases is represented. A is at lat. 27° N., long. 82° W., and B is at lat. 43° S., long. 127° E. Then angle AP_NB is $(180° - 127°) + (180° - 82°) = 53° + 98° = 151°$.

34. Distances on the Earth's Surface

Since the radii of all great circles on the earth are assumed to be equal (and equal to 3957 miles), the linear measure of any great-circular arc is known as soon as its angular measure is known. One minute of arc, for instance, on any great circle will always have a definite linear measure. Since this measure is of convenient size (slightly more than the ordinary statute or land mile and about 6080 feet), it is used as the unit of linear distance in spherical triangles on the earth's surface and is called the ***nautical mile***. Accordingly, if the side AB of the triangle ABP_N in Figure 129 were computed to be 132° 14′ 20″, its linear measure would be

$$(132 \times 60) + 14 + \frac{20}{60} = 7920 + 14 + \frac{1}{3} = 7934\tfrac{1}{3} \text{ nautical miles.}$$

This measure is usually considered entirely adequate. If for any reason the distance in land miles is desired, it can be found by multiplying the number of nautical miles by the number of land miles in a nautical mile, namely,

$$\frac{6080}{5280} = \frac{38}{33}.$$

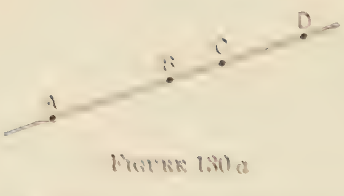


FIGURE 130 *a*

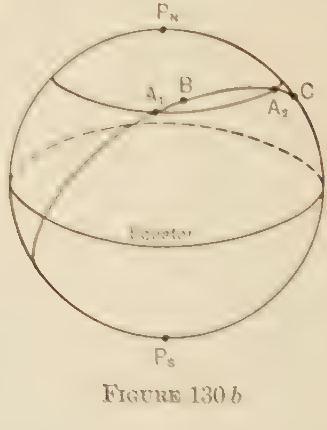


FIGURE 130 *b*

35. Directions on the Earth's Surface

Directions on the earth's surface, as well as positions and distances, are important. In plane trigonometry (see Figure 130*a*) the directions of all points from a given point, A, on a particular straight line through the point (and on the same side of the given point) are the same. The analogous situation does not obtain, in general, in spherical trigonometry. In Figure 130*b* A_1 and A_2 are two points having the same latitude. B and C lie on the great circle determined by A_1 and A_2 such that B separates A_1 and A_2, and A_2 separates B and C. Then B is obviously somewhat north of east from A_1, A_2 is exactly east of A_1, and C is somewhat south of east from A_1.

Because of the arbitrary convention of direction on the earth's surface, the equator and the meridians are special great circles on the earth. A point moving in a constant sense around the equator is forever moving in a constant direction (either east or west). A point moving in a constant sense around a meridian moves over half of the circle in a constant compass direction (either north or south) and then abruptly changes direction by 180° which direction it then maintains for the next half of the circle. On all other great circles on the earth's surface a point moving in a constant sense is *constantly changing direction.* For this reason to say that London is northeast of New York is not to say that in flying from New York to London on a great circle a plane will fly in a northeast direction. The plane will *start* in a northeasterly direction, be flying exactly east at some instant during the trip, and will *enter* London in a southeasterly direction. (See Figure 131.) To say that London lies to the northeast of New York is merely to say: (1) that its latitude is more northerly than New York's, and (2) that it is shorter to go from New York to London on a great circle by starting to the east (instead of to the west) of the meridian of New York.

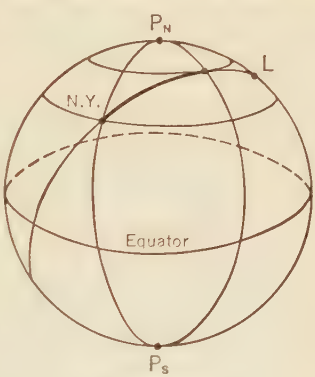


FIGURE 131

Actually it is too complicated to change direction perpetually on great-circle journeys. At sea it is customary to change as often as the computed amount of the proper change is large enough to be significant and to then maintain a constant course between changes. This will generally mean a change in direction once in from one to three or four hours depending on the course and speed of the ship. Obviously, in the case of flying these changes of direction must be made more frequently. The great-circle path is thus approximated by a series of arcs of constant direction. These arcs of constant direction are, in general, not even arcs of small circles. They are called ***rhumb lines*** or ***loxodromes***. If a plane should fly continuously on a rhumb line of direction anything but exactly north, south, east, or west, the plane would forever spiral

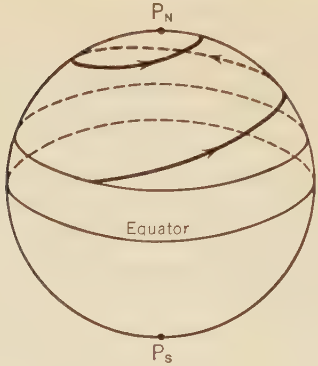


FIGURE 132

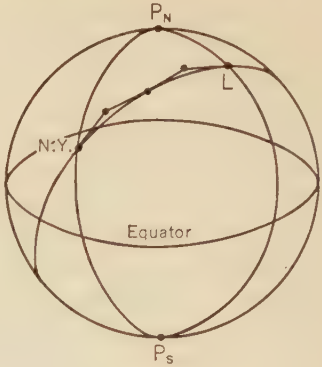


FIGURE 133

toward one of the poles, getting infinitely close to the pole but never quite reaching it. Figure 132 shows a rhumb line of direction slightly north of east from a point in the northern hemisphere. Figure 133 suggests how a ship sailing from New York to London approximates a great-circle course by a series of rhumb line arcs. If at one of the changes of direction, the direction set happened to be exactly east or exactly west, the subsequent arc sailed would be an arc of a small circle. By frequently altering direction and sailing on rhumb line arcs between changes of direction the actual great-circle distance between the two points is closely approximated. Furthermore, the direction set at each change of direction is certainly going to be very nearly, if not exactly, that of the great circle connecting the particular point of change of direction with the point of destination. For these two reasons great-circle distances and directions continue to be the basis of flying and sailing long distances.

Having agreed from the above that the direction of a general great circle depends on the particular point considered on the great circle, we will examine the convention of describing this direction at particular points. Figure 134 shows the fundamental spherical triangle ABP_N involved when two points on the earth's surface are given.

DEFINITION: *If a plane or ship is traveling in a given direction at a given instant, this direction is said to be the* **course** *of the plane or ship, and the plane or ship is said to be* **on this course** *at the instant.*

DEFINITION: *The* **initial course** *is the course on which a plane or ship leaves its starting point.*

DEFINITION: *The* **course of arrival** *is the course on which a plane or ship reaches its destination.*

If in Figure 134 angle P_NAB is 103° and angle P_NBA is 72°, we can describe the direction of B from A *at* A as 13° south of west and the direction of A from B *at* B as 72° east of north. In order to do away with the necessity of referring these angles to various points of the compass, the following ***compass bearing*** convention is devised. According to this convention, if a plane or ship is traveling *from* A *to* B, its Initial Course is said to be 257° (or 257° true). Imagine an arrow drawn *through* A showing the direction of travel at A. Its head will be towards B from A. Then 257° is the angle at A between the *northern* part of the meridian and the *head* of the arrow (representing the initial course at A) through A *measured clockwise from north through east.* If B is the destination and if an arrow is drawn *through* B (with its head away from A), then the angle from the *northern* part of the meridian to the *head* of this arrow of the final course, *measured clockwise through east* will be 252°. Accordingly, the course of arrival is said to be 252° (or 252° true). If the plane or ship were flying from B to A *the initial course* and *course of arrival* would be 72° and 77°, respectively. The following table shows compass courses for a plane or ship traveling from any point at a given instant in various directions:

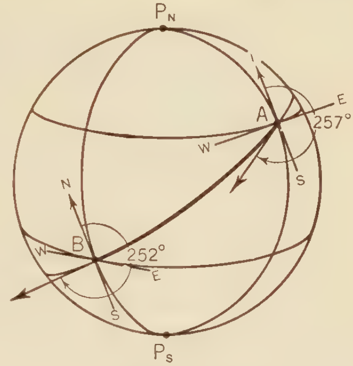


FIGURE 134

North	*East*	*South*	*West*	*South-(west)*	*10° West of North*	*South* 17° *East*
0°	90°	180°	270°	225°	350°	163°

36. Problems on Sections 32–35

1. Draw a large sphere and on it picture all the following points and indicate the given arcs and angles:

(*a*) Panama (Lat. 08° 58′ N., long. 79° 32′ W.).
(*b*) New York (Lat. 40° 40′ N., long. 73° 50′ W.).
(*c*) Juneau, Alaska (Lat. 58° 20′ N., long. 134° 35′ W.).
(*d*) Buenos Aires (Lat. 36° 30′ S., long. 60° 00′ W.).

2. Estimate the approximate distance in nautical miles and land miles (using the approximation 1 nautical mile = $1\frac{1}{7}$ land miles) between Sante Fe, New Mexico (lat. 35° 40′ N., long. 106° 05′ W.) and Casper, Wyoming (lat. 42° 51′ N., long. 106° 18′ W.). Sketch the figure.

3. If a highway runs due west from Philadelphia (lat. 39° 53′ N., long. 75° 10′ W.) to Columbus, Ohio, (lat. 39° 37′ N., long. 83° 00′ W.), approximately how long is it, neglecting hills and valleys?

4. In each case below an airplane is leaving the given point on the earth's surface on the given great-circle course. Draw a sphere and on it indicate the great-circle path of the airplane, indicating on the figure all given arcs and angles:

(*a*) Denver (lat. 39° 45′ N., long. 105° 00′ W.) on course north 60° west.
(*b*) Rio de Janeiro (lat. 23° 00′ S., long. 43° 20′ W.) on course 210° true.
(*c*) London (lat. 51° 25′ N., long. 00° 20′ E.) on course 120° true.

5. If an Eskimo travels due east on the Arctic Circle (lat. 66° 33′ N.) for 100 miles, by how much has he changed his longitude? If an airplane flies 100 miles due east on the equator, by how much has the plane's longitude been changed?

6. If a ship left the coast of Ecuador on the equator (long. 80° W.) and sailed due west, it would not meet any land of considerable size until it reached Halmahera Island (one of the Moluccas or Spice Islands in the East Indies) in longitude 128° E. How long a voyage would the ship have sailed?

7. If an airplane is to be flown on a great-circle course from Leningrad (lat. 59° 55′ N., long. 30° 20′ E.) to Seward, Alaska (lat. 60° 07′ N., long. 149° 20′ W.), approximately on what course should the plane leave Leningrad? What approximately would be the plane's course of arrival? If the plane can average 200 land miles per hour, how long will this flight take? If the plane consumes on the average 50 gallons of gasoline an hour, how many gallons of gasoline are saved by the plane's taking the great-circle course instead of flying continually due west?

37. Solution of Terrestrial Problems

The procedure to be followed in solving a terrestrial problem is as follows:

1. Draw a sketch of the earth showing the poles, equator, the Greenwich Meridian (conveniently placed for showing the particular given points), and the given points themselves. Show the meridians of the given points.

2. Draw the minor great-circular arcs between pairs of given points to complete spherical triangles whose vertices are one of the poles and a pair of given points. On each such triangle indicate parts whose measures can be easily calculated from the given positions of the points and the given directions by writing these measures on the proper parts and encircling them.

3. For each triangle needed in the solution make another sketch of this triangle by itself and removed from the earth's surface. Letter the triangle and show the known parts encircled.

4. By the methods discussed in the previous chapters solve only as much of the triangle (or triangles) as is required. Always reduce the problem to the solution of the simplest possible type of triangles, namely right, isosceles, quadrantal, or general oblique in this order.

5. Make sure that the answers are in the required form.

The following definition involves a useful concept of great-circle paths of the earth's surface:

DEFINITION: *A* ***vertex of a great-circle path on the earth's surface*** *is a point on the path nearest one of the geographical poles.*

Thus any great-circle path, not on the equator or a meridian, has two vertices: the north vertex, V_N, and the south vertex, V_S. (See Figure 135.) The vertices need not, of course, lie in between the two given points determining the particular great-circle path under consideration.

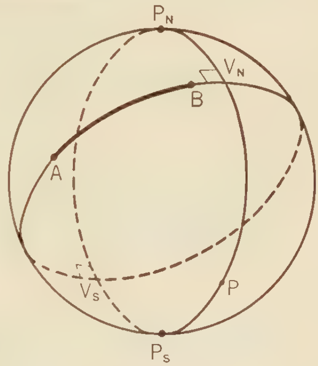


FIGURE 135

THEOREM: *The meridian of a vertex of a great-circular path is perpendicular to the path.*

The proof is accomplished merely by quoting Napier's Corollary 3.

EXAMPLE 12: New York and Naples have practically the same latitude. Assume they have exactly the same latitude (that of New York) and compute the approximate distance saved in flying from New York to Naples on a great-circle path instead of continually due east. (New York: lat. 40° 40′ N., long. 73° 50′ W.; Naples: lat. 40° 51′ N., long. 14° 26′ E.)

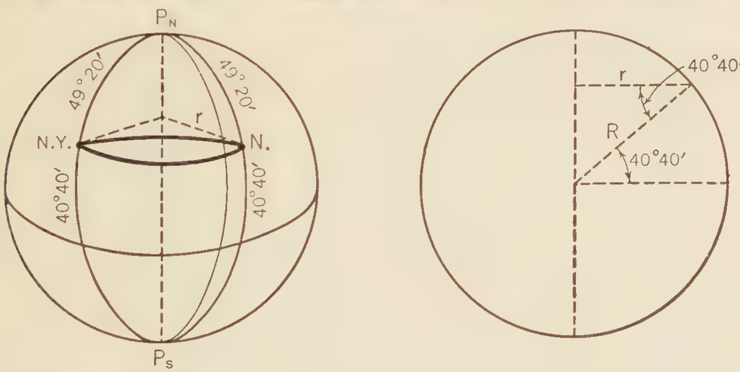


FIGURE 136

From Figure 136 the small-circle distance is given by:

73° 50′
14° 26′
87° 76′
2 | 88° 16′ = 88.27
44° 08′

$$r = 3957 \cos 40° 40' \text{ land miles} = \frac{33}{88} \cdot 3957 \cos 40° 40' \text{ naut. mi.}$$

$$= 3440 \cos 40° 40' \text{ naut. mi.}$$

$$s = r\,\theta = (3440 \cos 40° 40')\,\frac{88.27}{180} \cdot \pi = \textit{4020 nautical miles.}$$

(slide rule)

From Figure 137 the great-circle distance is given by:

$$\sin\frac{d}{2} = \sin 49^\circ 20' \sin 44^\circ 08'$$

$$\sin\frac{d}{2} = 0.528$$

$$\frac{d}{2} = 31^\circ 50' \quad \therefore\ d = 63^\circ 40'$$

$$d = 3820 \textit{ nautical miles.}$$

(*slide rule*)

$\therefore$ saving = 4020 − 3820 = **200 nautical miles.**

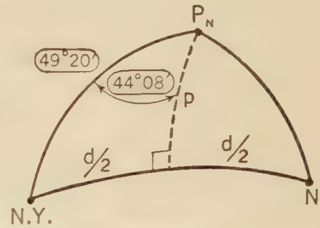


FIGURE 137

EXAMPLE 13: If a ship is to sail from Honolulu (lat. 21° 15′ 08″ N., long. 157° 48′ 44″ W.) to Sydney, Australia, (lat. 33° 51′ 41″ S., long. 151° 12′ 23″ E.) on a great-circle course, what must be its initial course and how many nautical miles must she sail? (See Figure 138.)

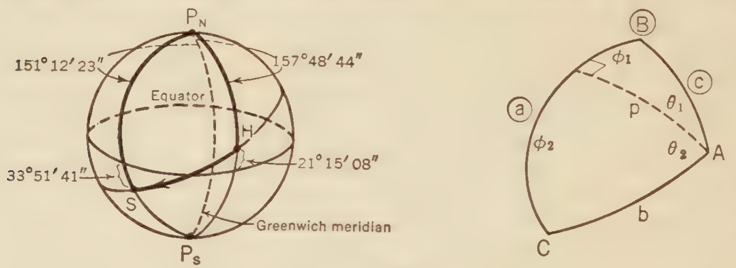


FIGURE 138

151 − 12 − 23	89 − 59 − 60
157 − 48 − 44	21 − 15 − 08
309 − 01 − 07	68 − 44 − 52 = c
359 − 59 − 60	
50 − 58 − 53 = B	90 − 00 − 00
	33 − 51 − 41
	123 − 51 − 41 = a

$$\sin p = \sin B \sin c \ (p \text{ in } I)$$

$$\tan \phi_1 = \cos B \tan c \qquad \cos B = \tan \phi_1 \cot c$$

$$\cot \theta_1 = \tan B \cos c \qquad \cos c = \cot \theta_1 \cot B$$

$$\phi_2 = a - \phi_1$$

$$\cot \theta_2 = \sin p \cot \phi_2 \qquad \sin p = \tan \phi_2 \cot \theta_2$$

$$\cos b = \cos p \cos \phi_2$$

$$A = \theta_1 + \theta_2$$

$$\text{Initial Course} = 359^\circ 59' 60'' - A$$

$$\text{Distance} = b \text{ in minutes}$$

The actual numerical evaluations are left to the student.

Answers: Course = 222° 18′ 32″
Distance = 4405.5 n. mi.

EXAMPLE 14: If a ship sails on a great-circle course from San Diego (lat. 32° 43′ N., long. 117° 10′ W.) to Cavite in the Philippine Islands, its course is 300° 40′. The position of Wake Island is lat. 19° 11′ N., long. 166° 31′ E.

a. How far from Wake Island will the ship be when in this island's longitude?

b. How close will the ship come to this island and when (with respect to the instant the ship reaches the longitude of Wake Island) will it be at this point nearest to Wake Island, if the ship makes 15 knots?

c. How close does the ship come to the north pole, what is its course at this point, and when is the ship at this most northerly point? (See Figure 139.)

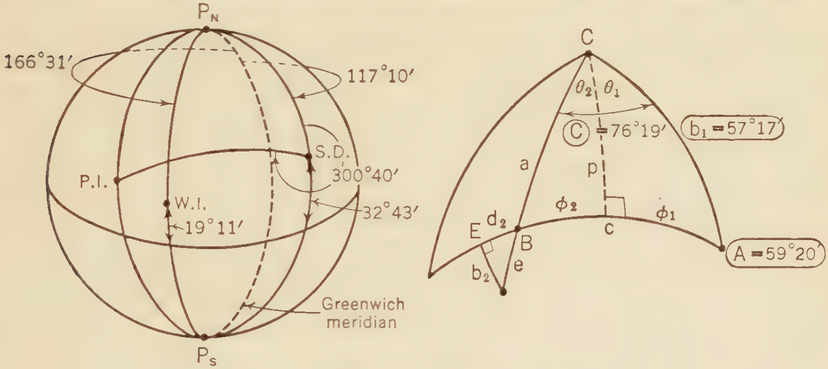


FIGURE 139

(a) $\sin p = \sin b_1 \sin A$, p in same quadrant as A.

$\cot \theta_1 = \cos b_1 \tan A$ $\qquad \cos b_1 = \cot \theta_1 \cot A$

$\theta_2 = C - \theta_1$

$\cot a = \cos \theta_2 \cot p$ $\qquad \cos \theta_2 = \tan p \cot a$

$\cos B = \sin \theta_2 \cos p$

$e = 90° - (19° 11') - a$

(b) $\sin b_2 = \sin B \sin e$

$\tan d_2 = \cos B \tan e$ $\qquad \cos B = \tan d_2 \cot e$

(c) $\tan \phi_1 = \tan b_1 \cos A$ $\qquad \cos A = \tan \phi_1 \cot b_1$

179° 60′
117° 10′
62° 50′

179° 60′
166° 31′
13° 29′
62° 50′
76° 19′

$C = 76° 19'$					
$b_1 = 57° 17'$	l sin 9.92498	l cos 9.73278			l tan 10.19219
$A = 59° 20'$	l sin 9.93457	l tan 10.22697			l cos 9.70761
$p = 46° 21' 35''$	l sin 9.85955		l cot 9.97938	l cos 9.83893	
$\theta_1 = 47° 39' 04''$		l cot 9.95975			
$\theta_2 = 28° 39' 56''$			l cos 9.94322	l sin 9.68097	
$a = 50° 04' 44''$			l cot 9.92260		
$B = 70° 40' 02''$	l sin 9.97479			l cos 9.51990	
$e = 20° 44' 16''$	l sin 9.54912			l tan 9.57820	
$b_2 = 19° 31' 10''$	l sin 9.52391				
$d_2 = 07° 08' 42''$				l tan 9.09810	
$\phi_1 = 38° 26' 54''$					l tan 9.89980

(a) $e = 20° 44' 16'' = 1200 + 44 + \frac{16}{60} =$ | $1244\frac{4}{15}$ nautical miles. |

(b) $b_2 = 19° 31' 10'' = 1140 + 31 + \frac{10}{60} =$ | $1171\frac{1}{6}$ nautical miles. |

$d_2 = 07° 08' 42'' = 420 + 8 + \frac{42}{60} = 428\frac{7}{10}$

$\therefore \frac{d_2}{15} = \frac{4287}{150} = 28\frac{87}{150}^h = 28^h + \frac{87}{15} \cdot 6^m =$ | $28^h\ 34^m\ 48^s$ later. |

(c) $p = 46° 21' 35'' = 2760 + 21 + \frac{35}{60} =$ | $2781\frac{7}{12}$ nautical miles. |

$\phi_1 = 38° 26' 54'' = 2280 + 26 + \frac{54}{60}$
$= 2306\frac{9}{10}$ | Course = 270° |

$\frac{\phi_1}{15} = \frac{23069}{150} = 153\frac{119}{150}^h =$ | $6^d\ 9^h\ 47^m\ 36^s$ out of San Diego. |

38. Problems on Chapter 4

1. A plane is flying a great-circle course across the Atlantic Ocean from New York (lat. 40° 40′ N., long. 73° 58′ 30″ W.). Sometime after leaving, the pilot, with sextant, observes his latitude to be 53° 17′ N. and estimates he has covered 1325 nautical miles. What is the pilot's longitude and what course should he be steering?

2. A pilot planning a great-circle flight across the Atlantic Ocean from New York (lat. 40° 40′ N., long. 73° 58′ 30″ W.) wishes, because of weather conditions, to stay south of 55° 30′ north latitude. He estimates he should allow 3° for error in his initial course (i.e., leave New York on a course 3° south of his most northerly allowable course). If he carries fuel for a flight of 3850 nautical miles, where would he be forced down because of fuel shortage?

3. Prove that for any great-circle course not on the equator nor through the poles, the vertices and points of intersection with the equator divide the entire great circle into four equal parts of length equal to one quadrant.

4. Let Q be one of the points of intersection with the equator of any great circle (not the equator). Let A and B be two points on this great circle such that the arcs AQ and QB of this great circle are equal. Prove that a ship sailing in a given sense on this great circle will have the same course angle at B as at A.

5. The Law of Sines for spherical triangles states that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Prove this law by dropping perpendiculars from two vertices of the triangle and then equating two expressions for each altitude from Napier's Rules.

6. Using the above Law of Sines, prove that for any ship or plane traveling on a great circle, not the equator nor a meridian, the course as it crosses the equator is either the most northerly or the most southerly course of all the courses on which the ship or plane can sail or fly on this particular great circle. (If the crossing of the equator is from the south to the north, the course at the crossing will be the most northerly; if the crossing is from the north to the south, the course at the crossing will be the most southerly of all possible courses on this particular great circle.)

7. A cruiser and a destroyer both leave Brisbane, Australia (lat. 27° 27′ 32″ S., long. 153° 01′ 48″ E.) at the same time to sail on a great-circle course to Acapulco, Mexico (lat. 16° 49′ 10″ N., long. 99° 55′ 50″ W.). The cruiser sails at 18 knots and the destroyer at 29 knots. When the two ships again have the same course what will this course be? (Cf. problem 4.)

8. A ship was known to have sailed from the Galapagos Islands, lying on the equator in longitude 90° 30′ W., on course 280° 30′ for the Pelew Islands. On Sunday, January 18, a wireless operator on Canton Island (lat. 02° 44′ N., long. 171° 45′ W.) picked up a message, dated Monday, January 19, from the ship, in which message the ship's latitude was given as 08° 18′ N. Less than half an hour later the Canton Island operator received a distress call from the ship, which then became silent. On what course should rescue craft have been sent out from Canton Island to the ship?

9. If a plane leaves Oslo, Norway (lat. 60° N., long. 10° 42′ E.) on a great-circle flight, what is the plane's latitude and longitude after flying 2700 nautical miles, if its initial course was due east? (Evaluate arc functions by slide rule or tables of natural functions.)

10. In each of the following pairs of cities the latitudes of the two cities in each pair are approximately the same. Assuming both cities have the latitude of the first named city compute the distance saved in flying between the two on a great-circle path instead of a course constantly due east or due west. Compute the course of departure and arrival in flying from the first to the second city and also the highest latitude reached. Use either slide rule or logarithms.

(*a*) Portland, Ore. (Lat. 45° 38′ N., long. 122° 45′ W.)
Montreal, Canada (Lat. 45° 33′ N., long. 73° 35′ W.)
(*b*) Calcutta, India (Lat. 22° 30′ N., long, 88° 30′ E.)
Hong Kong, China (Lat. 22° 16′ N., long, 114° 12′ E.)
(*c*) Moscow, Russia (Lat. 55° 45′ N., long. 37° 36′ E.)
Edinburgh, Scotland (Lat. 55° 55′ N., long. 03° 10′ W.)
(*d*) Bern, Switzerland (Lat. 46° 56′ N., long. 07° 23′ E.)
Quebec, Canada (Lat. 46° 53′ N., long. 71° 20′ W.)

11. If a plane flies — or a ship sails — from the first point to the second point of each of the following triples of points, will the plane or ship pass to the left or to the right of the third point? How close (in nautical miles) will the plane or ship come to the third point?

(*a*) Washington, D.C. (Lat. 38° 55′ N., long. 77° 00′ W.)
Denver, Colo. (Lat. 39° 45′ N., long. 105° 00′ W.)
Indianapolis, Ind. (Lat. 39° 45′ N., long. 86° 13′ W.)
(*b*) Rome, It. (Lat. 41° 45′ N., long. 12° 15′ E.)

Jerusalem, Pal. (Lat. 31° 46′ N., long. 35° 14′ E.)
Athens, Grc. (Lat. 37° 54′ N., long. 23° 52′ E.)

(*c*) London, Gt. Brit. (Lat. 51° 25′ N., long. 00° 20′ E.)
Moscow, Sov. Un. (Lat. 55° 45′ N., long. 37° 36′ E.)
Hamburg, Ger. (Lat. 53° 33′ N., long. 10° 02′ E.)

(*d*) London, Gt. Brit. (Lat. 51° 25′ N., long. 00° 20′ E.)
Rome, It. (Lat. 41° 45′ N., long. 12° 15′ E.)
Geneva, Switz. (Lat. 46° 13′ N., long. 06° 11′ E.)

(*e*) Halifax, Can. (Lat. 44° 40′ N., long. 63° 45′ W.)
Nassau, Ba. Is. (Lat. 25° 04′ N., long. 77° 22′ W.)
Cape Hatteras, N.C. (Lat. 35° 15′ N., long. 75° 31′ W.)

(*f*) Paris, Fr. (Lat. 48° 50′ N., long. 02° 20′ E.)
Stalingrad, Sov. Un. (Lat. 48° 40′ N., long. 44° 30′ E.)
Krakow, Pol. (Lat. 50° 05′ N., long. 19° 59′ E.)

(*g*) Bordeaux, Fr. (Lat. 44° 40′ N., long. 00° 30′ W.)
Sevastopol, Sov. Un. (Lat. 44° 45′ N., long. 33° 32′ E.)
Venice, It. (Lat. 45° 26′ N., long. 12° 20′ E.)

(*h*) Budapest, Hung. (Lat. 47° 30′ N., long. 19° 05′ E.)
Bordeaux, Fr. (Lat. 44° 40′ N., long. 00° 30′ W.)
Geneva, Switz. (Lat. 46° 13′ N., long. 06° 11′ E.)

12. Stations A and B on the earth's surface are 1000 nautical miles apart. Airplanes One and Two each leave station A on a great-circle flight whose path makes an angle of 37° 20′ with the great-circle path from A to B, plane Two leaving sometime after plane One and both flying at 200 nautical miles per hour. Sometime after both planes have left station A both planes are ordered by radio to change course immediately and fly on great-circle course to station B. If both airplanes arrive at station B three and one half hours after the order to change course was given, what was the total flying time for airplane Two in flying from A to B?

13. During a fog a ship asks a radio direction-finder station at St. John's, Newfoundland (Lat. 47° 32′ N., long. 52° 40′ W.) for a bearing. The station responds that the ship's bearing at the station is 37° 25′. By dead reckoning the navigator of the ship feels fairly certain that his longitude is 50° 30′ W. What then is the ship's latitude? (Note that radio waves follow great circles. Cf. Appendix III, section 37.)

14. A ship in a fog off the Carolinas asks radio direction-finder stations at Cape Hatteras (Lat. 35° 15′ N., long. 75° 31′ W.) and at Cape Fear (Lat. 33° 51′ N., long. 77° 58′ W.) for radio bearings. These stations respond that the ship's bearings at the stations are: at Cape Hatteras, 200° 25′; at Cape Fear, 76° 12′. Find the ship's position by the solutions of spherical triangles. (Note that radio waves follow great circles. See Appendix III, section 37, for the method actually used in practice to determine position from two radio bearings. The method suggested here is theoretically sound but requires more time than can usually be spent on this problem at sea.)

CHAPTER 5

Celestial Applications

39. The Program

All terrestrial application problems presuppose as given the positions of the particular points on the earth's surface mentioned in the problems. Such data in practice are obtained by means of celestial observations. Before systematizing celestial observations numerically, it is well to review their characteristics in general as experienced by casual observation.

In this review the emphasis will be placed on the general nature and fortunate regularity of these complex celestial phenomena. The many bedeviling small corrections which must be applied to observations of some of these phenomena will be dealt with later on. Such corrections by a professional navigator are frequently matters of the lives or deaths of many people. But for the student being introduced to celestial applications of spherical trigonometry, such corrections will warrant, in general, only footnote mention. Furthermore, the general and unparticularized concept of time will suffice for the present. In a later section it will be adequately systematized.

PART ONE: *Description and Explanation of Celestial Phenomena*

40. The Fixed Stars

One of the most fortunate instances of order in the universe is the apparent fixity of all but a dozen * of the myriad heavenly bodies. On any cloudless night each star — that is, any heavenly body with the above noted dozen or so exceptions — is observed to move in a particular circular arc about a point — as center — in the sky very close to a certain star called the "north star" or "Polaris," while at the same time keeping its position relative to all the other stars unchanged. Such heavenly bodies — that is, all but the dozen or so exceptions — are therefore called the Fixed Stars. These are known to be at tremendous distances from the earth.†

* The sun, moon, planets, and comets.

† See note on page 116.

Two descriptions of the phenomenon of the fixed stars are possible: In both cases the stars are conceived as permanently set in a gigantic sphere of transparent material, offering no resistance, with the earth as center. Then the observed motion of these stars can be described as due either to the rotation about a fixed axis of this sphere of stars while the earth stays motionless, or to the rotation of the earth about a fixed axis while the sphere of stars stays motionless. The second explanation, because of its consistency with many other observed phenomena, is the one adopted. Accordingly, *The earth is said to rotate about a particular diameter (called its axis) which maintains a constant direction.* Whatever other motion the earth may have, it so moves that the direction of its axis at any time is parallel to the direction of its axis at any other time. The north star in the fixed heavens is very nearly in line with

FIGURE 140. URSA MAJOR, OR THE BIG BEAR

FIGURE 141. ORION

FIGURE 142. CASSIOPEIA

this constant direction of the earth's axis and therefore very closely points the direction of the north terrestrial pole.

Precisely because of the fixity of the fixed stars the ancients, observing that they formed set patterns in the sky, made a rough division of the heavens by distinguishing several groups of stars or "constellations." Figures 140 to 142 picture the principal stars in some of these constellations. A familiarity with several of these constellations enables one to roughly designate any desired part of the heavens.

41. The Sun and the Earth's Orbit

To a person on the earth the most spectacular star, which, like all the other stars, periodically reappears because of the rotation of the earth about its own axis, is that star which we call the sun. So complete is the dependence of the earth and all its inhabitants upon the sun, and so obvious is its influence on human activities, that, among other things, our generally used system of time is based on this star.

In section 52 the subject of time will be treated in detail. It is enough here to recall that in ordinary usage the length of a day of twenty-four hours is the time interval between two noons, where a noon is the instant at which the sun is highest in the sky for that day.* Now if the sun be like all the other stars, we would expect that any other star could equally well define the length of a day, that is, as the time interval between two successive appearances of this star at its highest point in the sky. This is true, but, as can readily be observed, the length of such a day would be shorter: We shall call the length of a day as determined by the sun a ***solar day*** and, analogously, the length of a day as determined by any other star a ***sidereal day***. That our clocks and watches indicate fractional parts of solar days is a natural inference from the observation that, day after day and year after year, the sun appears to be *approximately* at its highest point when our time pieces indicate twelve o'clock noon. Now it can easily be observed that a particular star will be highest in the sky a month from tonight about *two hours earlier* than for tonight, indicating that a sidereal day is about four minutes shorter than a solar day. Further qualitative evidence of this difference is obtained by observing the stars or constellations in which the sun rises or in which it sets from day to day, that is, by noting the stars last visible on the eastern horizon before sunrise and the stars first visible on the western horizon after sunset. Accordingly, the sun

* That this is not strictly the case will be seen in section 52. But the variations from this approximation can at this point readily be observed to be periodic rather than continuously accumulative.

appears to lag behind the stars by about the same four minutes a day.

Since four minutes a day amounts to about one day a year, there is one more sidereal than solar day per year. In other words, the earth, while rotating once a day on its axis, behaves with respect to the sun in some special manner different from its behavior with respect to all the thousands of other stars, and, furthermore, the unit of time called the year must be the unit of time which has been adopted precisely because of and as an exact measure of this special behavior of the earth with respect to the sun but not with respect to all the other stars. ***In addition to rotating on its axis the earth revolves in an orbit about the sun but about no other star.*** This *revolving* exactly accounts for the difference (that is, one day a year) between the earth's *rotation* as shown through the reappearance of the sun and as shown through the reappearance of all the other stars.

A homely example, pictured in Figure 143, will be instructive here: Consider a single pine tree in the center of a large clearing in a forest of birch trees. Let a man walk around the pine tree, keeping very much closer to this pine tree than to the nearest birch tree. If he walks around so as to be always facing the pine tree (necessitating walking sideways for the whole circuit), he will have faced exactly once, and turned his back on exactly once, every birch tree. Now let the man walk around the pine tree while always facing in some one direction, say west (necessitating walking alternately forward, sideways, and backward). He will then have faced the pine tree once and have had his back to it once, *but he has not appreciably changed his aspect toward any one of the birch trees.** In each of these two cases there is a *difference of exactly one* between the number of complete changes of aspect of the man toward the pine tree on the one hand and toward all the birch trees on the other, and this difference has been due solely to the difference in behavior of the man with respect to the pine tree and with respect to all the birch trees, that is, he has made a circuit around the pine tree but not about the birches.

FIGURE 143

Now, if the man spins around as he makes his circuit of the pine tree, the difference in the number of complete changes of aspect of the man toward the pine tree on the one hand and toward the birch trees on the other will always be exactly one regardless of the number of spins per circuit or the direction of spin with respect to the direction of the circuit. *When the spin*

* We assume that the clearing is very large in comparison with the man's circuit of the pine tree.

*is "**with** the circuit" (that is, in the same direction as the circuit — clockwise or counter-clockwise), in each complete circuit of the pine tree the man faces each birch tree exactly one **more** time than he does the pine tree; when the spin is "**against** the circuit" (that is, clockwise when the circuit is counter-clockwise or vice versa), in each complete circuit of the pine tree the man faces each birch tree exactly one **fewer** times than he does the pine tree.*

FIGURE 144

(a) Two spins per circuit *with* the circuit:
One change in aspect with respect to the pine tree.
Two changes in aspect with respect to the birches.

(b) Two spins per circuit *against* the circuit:
Three changes in aspect with respect to the pine tree.
Two changes in aspect with respect to the birches.

Figures 144 *a* and *b* picture these relations for some simple cases. The man is represented by a short segment with a dot at one end indicating his face.

The application of the above example of pine and birch trees to solar and sidereal days is immediate and explains their difference: The sun replaces the pine tree, the earth the man, and the fixed stars the birch trees. Accordingly, we conclude, by way of explaining the difference between solar and sidereal days, that *the earth, in addition to rotating on its axis, revolves in an orbit about the sun but about no other star.* Because the number of sidereal days per year has been observed to be one *more* than the number of solar days (and therefore 366 sidereal days = 365 solar days = 1 ordinary year), we conclude that *the direction of the earth's rotation on its own axis is with the direction of revolution in its orbit about the sun.* The unfortunate fact that the number of sideral days (and, therefore, likewise the number of solar days) in a year is not integral (that is, the number of times that the earth rotates about its axis between successive reappearances of the earth in a given position in its orbit about the sun is not a whole number) necessitates leap years. The fact that leap years occur in general once in four years indicates that the proper fraction of a sidereal day over 366 must be about one-fourth. This in fact is the case.*

In this review of the motions of the earth two more observations should be explained: the climate zones and the seasons. These phenomena are

* 1 year = 365.2422 mean solar days = 366.2422 sidereal days.

related, and both are brought forcefully to the attention of even the casual observer. The fact that many regions of the earth each year experience a season of very warm weather followed by one of very cool weather obviously could never completely be explained by any eccentric position of the sun in the earth's orbit, causing the earth to come much closer to the sun during the hot season than during the cold season,* since the hot seasons do not occur at the same times of the year for all those points on the earth which experience hot and cold seasons. The fact that pairs of points such that it is hottest for one when it is coldest for the other, and vice versa, lie on opposite sides of the equator is important. The equator and the poles are significant in the phenomena of climate zones: The region around the equator is always very warm, experiencing little variation in temperature. In general, the farther the region is away from the equator, the lower is the annual mean temperature and the greater the variation in the seasons. Consequently, we must conclude that *the earth so moves in the course of a year as to change the aspect of its equatorial regions relative to the sun by very little, while at the same time changing the aspects of its polar regions relative to the sun very markedly and alternately for the two hemispheres.* Now this changing of aspects relative to the sun of the polar regions cannot be accounted for by any change in direction of the earth's axis, for we have remarked (section 40) on the evidence of the fixity of this direction.

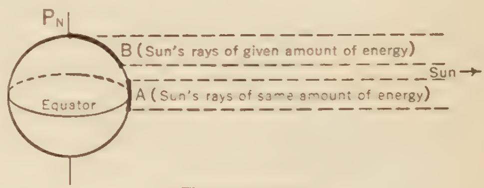


FIGURE 145

A and B represent the same amount of heat and energy from the sun, as each band of rays is of the same width. The arc of the earth's surface over which rays A are spread is obviously smaller than the arc over which rays B are spread. Since the same amount of heat in the case of rays A is spread over a smaller area than in the case of rays B, the regions of the A rays are warmer than those of the B rays.

This naturally suggests that we consider *what this fixed direction is relative to the sun.* If the fixed direction of the earth's axis were perpendicular to the plane of the earth's orbit, there would exist climate zones. The zone about the equator, receiving the sun's rays more nearly normal to the earth's surface than regions near the pole (see Figure 145), would receive a larger amount of heat per unit area than the polar regions. The equatorial zone would therefore be torrid and the polar regions frigid. But, since no region on the earth would ever change its aspect

* The opposite is true for northern latitudes, as the sun is at perihelion or nearest the earth in January (January 2 for 1943).

relative to the sun during the year (except for the daily night-and-day change), there could be no seasons for any region of the earth. But, if the axis of the earth were not perpendicular to the plane of its orbit, it could still be fixed in direction. Since this would immediately account for our seasons, as Figure 146 shows, we adopt this assumption. Measured observations verify this assumption and set the angle at which the plane of the earth's equator — perpendicular to the fixed direction of the earth's axis — is inclined to the plane of the earth's orbit at about 23° 27′.

DEFINITION: *The angle which the fixed * plane of the earth's equator makes with the fixed † plane of the earth's orbit is called the* ***obliquity of the ecliptic*** *and is approximately 23° 27′.*

The poles in the heavens are considered at infinite distances from the earth. Consequently, the earth's axis continues to point to these imaginary points in the heavens without having to tilt, even though this axis in the course of a year is displaced nearly 200 million miles. The star Polaris is so close to the north pole in the heavens and is so remote (67 light years) from the earth that the 200-million-mile displacement of the earth's axis due to the earth's orbital motion changes the angle which the line from the earth's center to Polaris makes with the earth's axis by about one-tenth of one second of arc.

We began our discussion of observations of the sun by noting one, and the less conspicuous, aspect of its *apparent* lack of fixity as a star: namely, its appearance each day at sunrise a little behind the group of stars in which it rose on the preceding day. We explained this lack of fixity, accounting for the four-minute discrepancy between solar and sidereal days, by the earth's revolution about the sun. We now see that this revolution, plus the tilting of the axis of rotation about the sun, accounts immediately for the more conspicuous aspect of the sun's lack of fixity: its being at progressively *different heights above the horizon at noon* for any given point of observation according to a periodic variation of one year's cycle. In Figure 146 point A is some point on the tropic of Cancer,‡ and B is the point of intersection of the tropic of Capricorn § and the meridian of A. Then a person at A sees the sun at noon directly overhead on June 21 but more than 45° below the zenith on December 21. (Just the opposite, of course, is true for a person at B.)

By way of summarizing our principal experiences of the universe

* Very nearly fixed. The axis wobbles very slightly, because earthquakes and the melting of huge masses of ice change the center of mass slightly.

† Very nearly fixed. The uneven attraction of the earth's moon and the planets on the earth tilts this orbital plane very slightly.

‡ That is, in latitude 23° 27′ N.

§ That is, in latitude 23° 27′ S.

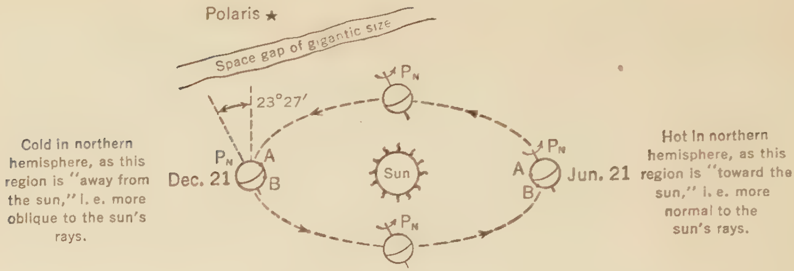


FIGURE 146

about us and the consequent explanations of the earth's behavior in this universe, let us recall that:

1. *We explain night and day and the* **apparent** *rotation of the fixed stars about Polaris by the earth's rotation about an axis of direction fixed in space and pointing very nearly in the exact direction of the star Polaris, which, with practically all the other heavenly bodies, forms a universe of stars fixed in space and at gigantic distances from the earth.*

2. *We explain that aspect of lack of fixity of the sun which is manifested by the sun's taking about four minutes longer to reappear each day than do the other stars by the* **revolution** *of the earth in an orbit about the sun in the same sense as the sense of its* **rotation** *about the axis — thus making slightly more than 366 sidereal days in a year of slightly more than 365 solar days.*

3. *We explain that aspect of lack of fixity of the sun which is manifested by the sun's periodic (period equal to one year) change in daily maximum height in the heavens, for any particular observation point, by the assumption that the earth's fixed axis of rotation tilts from the normal to the plane of the earth's orbit, the plane of the ecliptic, by about 23° 27′.*

Ironically enough, despite all our efforts to explain (as summarized in 2 and 3 above) the two aspects of marked lack of fixity of the sun, it is really just about as fixed as all the other stars for which we reserve the term "fixed stars." The pine tree in the example of page 112 was just as fixed as the birches. The earth, not the sun, is the cause of this *apparent* lack of fixity of the sun. The earth happens to be very much closer to the sun than to any other star,* and, furthermore, the earth behaves toward the sun in a

* The star next nearest to the earth after the sun is Proxima Centauri. Its distance from the earth is listed below, along with the distances from the earth of some other prominent stars. (DEFINITION: *A* ***light year*** *is the distance that light travels in one year. Since the speed of light is 186,000 miles per second, one light year is about 5.87 trillion miles).*

	Distance from Earth
The Sun	0.000016 light years
Proxima Centauri	4.17 light years
Sirius	8.3 light years
Vega	50. light years
Polaris	67. light years
Arcturus	140. light years
Andromeda Nebula	1,000,000. light years

unique way: it revolves about it but about no other star. But, since we are primarily interested only in celestial relations existing between the other heavenly bodies on the one hand and the *earth* on the other, we shall continue to bar the term "fixed star" from the sun.

42. Problems on Sections 40-41

1. Describe the changes in phenomena observable from the earth, if the earth's axial rotation were kept at the same speed with the direction of the earth's axis unchanged but with the sense of the axial rotation reversed.

2. What conditions would obviate the necessity for leap years? Is every fourth year a leap year? Why?

3. Assuming the sun were fixed in space,* compare the speed of a given point — not a pole — on the earth's surface at different times of the day. What points on the earth's surface move fastest and when? What points move slowest and when? What points on the earth's surface have most nearly constant speeds?

4. If the earth revolved in its orbit with no axial rotation, what could be said about the need for a system of leap years?

5. Describe the changes on the earth's surface which would result if the earth's axis were fixed but fixed in the plane of the earth's orbit.

6. How can the tropical zone on the earth's surface be defined in terms of the position of the sun in the sky at noon? Upon what does the width of the tropical zone depend? How narrow might the tropical zone be, granting rotation about a fixed axis and revolution in a plane orbit about the sun?

7. If the phenomena of day and night, the motion of the fixed stars, and the succession of seasons on the earth were accounted for by a fixed earth with the fixed stars and the sun rotating about the earth, what would then be the nature of the motion of the sun? Show that such an assumption would set the sun apart from the other stars in reality instead of only in appearance. Point out the obvious difficulties in interpreting the phenomena of day and night on the other planets under such an earth-center theory.

PART TWO: *Co-ordinate Systems in the Heavens*

43. Introduction to Co-ordinate Systems in the Heavens

The previous sections in this chapter have described the facts of the earth's behavior with respect to the sun and the fixed stars. We are, therefore, free to introduce systems of co-ordinates for measuring those celestial observations which will provide data (such as the latitude and longitude of points on the earth's surface) essential to the solution of triangles on the earth's surface. For the actual measuring of such observations, three things are necessary: some instrument, such as a sextant, transit, etc.,† designed to measure angles between observed points;

* The speed of the sun in its orbit about the center of the galaxy which projects on the celestial sphere as the "Milky Way" is about 200 miles per second.

† See Appendix III.

a timepiece; and a copy of the current edition of either the *Nautical Almanac* or the *Air Almanac*. These government publications tabulate various space co-ordinates which the principal heavenly bodies will possess at certain times during the current year. Such tabulations are computed on the basis of careful observations of these bodies over a period of many years. Brief descriptions of, and excerpts from, the *Nautical Almanac* and the *Air Almanac* are given in Appendix IV. In the text which follows mention will explicitly be made as to just what quantities are to be obtained from an almanac. Furthermore, practice in actually obtaining specifically required data from these almanacs will be provided in some of the problems.

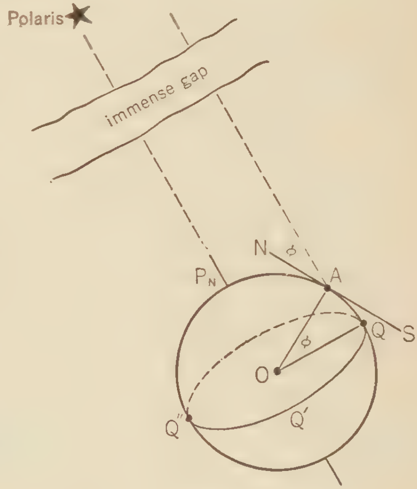


FIGURE 147

A simple example of how terrestrial data can be obtained from celestial observations should offer sufficient promise of the usefulness of the extensive systems of co-ordinates which follow. *For an observer in the northern hemisphere the angular distance of Polaris above the observer's northern horizon is his latitude north of the equator.** In Figure 147, A is the point of observation on the earth's surface. NAS, tangent to the earth at A, represents the observer's horizon, and $QQ'Q''$ is the equator. Because the distance from the earth to Polaris is so immense in comparison with the earth's radius, the line of sight of Polaris at any point (such as A) on the earth's surface can be considered parallel to the earth's axis. Therefore, the angles marked ϕ, having sides respectively perpendicular, are equal. But angle QOA is the latitude of A, and the other angle ϕ is the "altitude" of Polaris, or its angular height above the northern horizon.

Before describing any system of celestial co-ordinates by which celestial observations are to be systematized it is essential to develop the domain to which these co-ordinates are to apply. This involves reducing the three dimensional space about the earth to the two dimensional concept of the "celestial sphere."

* That is, when the corrections for the very slight angle between the earth's fixed axis and the line from the center to Polaris, and also the corrections for "parallax," "dip," and "refraction" are made. Further mention of these corrections will be made below.

44. The Celestial Sphere

To a person with an unobstructed view on a cloudless night the heavens appear as a huge hemispherical vault with himself as center. Across this vault the fixed stars appear (because of the earth's axial rotation from west to east) to move in small circles from east to west with great regularity. Other celestial bodies are seen to move across this vault but with much more complex motions. The observer's experience that the heavenly bodies on the vault above him appear and disappear suggests, instead of a celestial vault, a ***celestial sphere***. The celestial sphere will accordingly be constructed as the domain for later co-ordinate systems.

If the center of the celestial sphere were considered at the observer's point of observation on the earth's surface, then all systems of co-ordinates that are subsequently set up on this celestial sphere would necessarily be dependent upon this particular point of observation. Such a restriction would be unfortunate and is accordingly avoided by considering the earth's center as the center of the celestial sphere.

As will later develop, it is desirable to have some systems of celestial co-ordinates dependent on the point of observation, but it is also essential to have one system independent of this. It is by comparing observations referred to a celestial co-ordinate system *dependent* upon the point of observation to co-ordinates based on a system *independent* of the point of observation that the position of the point of observation can be calculated.

An immediate advantage of considering the celestial sphere's center at the earth's center is our ability to imagine the meridians and other reference lines on the earth's surface projected from the center onto this celestial vault. The only disadvantage is one of the small corrections previously mentioned as essential to the navigator but not to the beginning student or layman.

DEFINITION: *The* ***celestial sphere*** *is an imaginary spherical shell which is concentric with the earth, whose radius is indefinitely large, and upon which are projected from the center all heavenly bodies as well as the earth's poles and equator.*

DEFINITIONS: *The* ***celestial poles*** *are the projections of the terrestrial poles upon the celestial sphere. The* ***celestial equator*** *or the* ***equinoctial*** *is the projection of the terrestrial equator upon the celestial sphere.*

The following enumerated properties of the celestial sphere are immediate consequences of its definition:

1. The terrestrial observer, practically at the center of the celestial sphere, sees this sphere as the inside of a spherical shell. He sees about

half of it at any given instant, and, because of his rotation with the earth, he may, in the course of the day, see portions of the whole sphere varying from just half, if he is at a pole of the earth, to the whole celestial sphere, if he is on the earth's equator.

2. Instead of seeing the heavenly body at any instant "where it is in space," he thinks of seeing it against the "backdrop" of the spherical concave behind the body. That is, he thinks of seeing not the body itself but its *projection* on the celestial sphere.

Accordingly, for each of the thousands of fixed stars the observer imagines a light has been firmly imbedded in the inside surface of the celestial sphere at the point in which this celestial sphere — of indefinitely large radius — is pierced by the extension of the ray joining the center of the earth to the fixed star so represented. One of these fixed stars — a not very conspicuous nor intrinsically outstanding one — is called Polaris. But, because the earth's axis of west-to-east rotation happens to be fixed in a direction almost in line with this star, a terrestrial observer to whom Polaris is visible *seems* to see this spherical concave, imbedded with fixed lights, rotating about Polaris once a day from east to west. However, the less egocentric and more useful notion of the fixity of the celestial sphere should be cultivated. The phenomenon of any other celestial body, such as a comet or a planet, whose position relative to these fixed stars changes, will be thought of as that bright spot on the celestial backdrop — celestial sphere — which is produced by a spotlight at the center of the earth as the spotlight is kept trained on the moving body imagined transparent. Hereafter, for the sake of brevity, a celestial body will sometimes be spoken of as "lying on the celestial sphere" when, in reality, it is this body's projection which is on the celestial sphere.

Figure 148 shows the celestial sphere with its center at the center of the earth. In order to make the earth of finite diameter distinguishable at the center of the celestial sphere of indefinitely large diameter, the scale has been ignored. A few constellations of fixed stars are shown on this fixed celestial sphere by means of fixed points to be thought of as obtained by projecting the actual fixed stars onto this sphere from the center. The "celestial equator" is shown on the celestial sphere as the imaginary great circle which is the projection of the earth's equator. The curved arrow around the earth's axis indicates the direction of the earth's axial rotation which gives the *illusion* that this celestial sphere (with its fixed stars embedded in it) rotates about the same axis once in slightly less than twenty-four hours.

3. Since no value is ever assigned to the radius of the celestial sphere, linear distances on it have no significance. Distances on the celestial sphere are always considered in terms of angular measures of arcs.

4. The observer actually sees the heavenly body at any instant on the backdrop of the celestial sphere at the point at which his line of sight to the body hits this backdrop. But for computation purposes the *recorded* position of this observed body on the celestial sphere is the posi-

FIGURE 148

tion as viewed from the earth's center (the center of the celestial sphere). The slight correction to be applied to the *observed* position to give the *recorded* position is called the correction for geocentric parallax, or, more commonly, ***parallax***. When the observed body is on the horizon this parallax correction is a maximum and is called ***horizontal parallax***. Figure 149, in which the scale is ignored, illustrates the parallax correction.

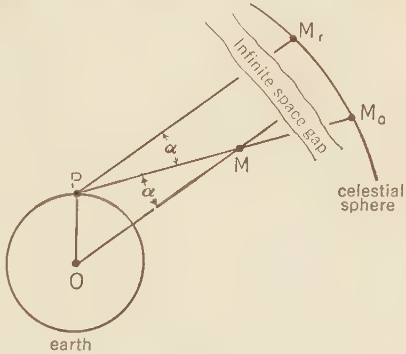


FIGURE 149

M is a celestial body in its actual position. *PM* is the line of projection of *M* onto the celestial sphere from the point of observation *P* on the earth's surface. Hence, the projection (M_o) of *M* on the celestial sphere is *seen* in this direction PM. But the *recorded* direction of the projection of M on the celestial sphere is the direction PM_r, where, because of the infinite radius of the celestial sphere, PM_r is parallel to OM, the line of projection from the *center* of the earth and celestial sphere. The angle α between these two directions — the observed and the recorded directions of M — is the correction for parallax and is obviously a maximum when PM is perpendicular to PO or when the observed body is "on the horizon." For fixed stars this correction is obviously infinitesimal. For the sun, moon, and planets it is small, but large enough to require consideration. The *Nautical Almanac* tabulates these parallax corrections. The *Air Almanac* tabulates only the moon's parallax.

5. As the earth revolves in its orbit, it pulls the center of the celestial sphere around with it. This produces a periodic displacement in the center of the celestial sphere of maximum amount about 186 million miles (the diameter of the earth's orbit). Since the radius of the celestial sphere is considered indefinitely large, this displacement in the center of the celestial sphere will not be considered to displace its surface. But the projections of the celestial bodies onto this celestial sphere may very well be materially altered. The degree of change in these projections, due to this annual change in the position of the center of the celestial sphere, varies with the type of celestial body whose projection is being considered:

a. For fixed stars the effect is insignificant.

The distances from the earth's center to the fixed stars are immense in comparison even with 186 million miles. Figure 150, in which the scale is ignored, illustrates the maximum shift in the projection of Arcturus on the celestial sphere for the center of projection at opposite ends of the earth's orbit. This shift amounts to about $\frac{1}{20}$ of one second of arc.

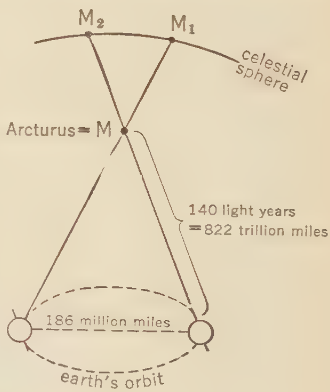


Figure 150

Consequently, for any system of co-ordinates which may later be set up on the celestial sphere the co-ordinates of the fixed stars will not suffer any appreciable annual change due to the motion of the center of the celestial sphere as the earth revolves in its orbit.

b. For the sun, moon, and planets the effect is considerable and is accordingly accounted for in the changing values of the celestial co-ordinates (to be discussed later) of these celestial bodies. In the case of the sun this effect is at the same time most significant and yet most difficult to experience correctly. Accordingly, this phenomenon — the effect of the revolution of the center of the celestial sphere (as the earth revolves in its orbit) upon the projection of the sun on the celestial sphere — will be the subject of the next section.

45. The Ecliptic or the Path of the Sun's Projection on the Celestial Sphere

The projection of the sun on the celestial sphere is strikingly different from the projection of any of the other stars for the reason that the earth revolves about the sun but about none of the other stars.

The earth, in its plane orbital motion, carries the center of the celestial sphere (the center of projection of all celestial bodies onto the celestial sphere) completely around the sun once every year. The result on the fixed sun's projection should therefore be a circle on the celestial sphere. This is precisely the case. The observer's actual experience of this circle, however, is unfortunately indirect. The earth's daily axial rotation gives to the sun a totally misleading apparent motion across the whole visible sky between every sunrise and following sunset. If the brilliance of the sun did not blot out all the stars during the day, the observer's experience of the circular path of the sun's projection on the celestial sphere would be much more direct. Imagine, for instance, a total eclipse of the sun lasting for a whole day. Then both the sun and some of the stars would be visible at the same time. Because of the earth's axial rotation the celestial sphere of the fixed stars would appear to have much the same east-to-west spurious rotation which the earth's axial rotation gives to the sun. But the sun would appear to lag behind these fixed stars by about one half degree of arc during each twelve hours of daylight, or by just that amount which accounts for the difference between a solar and a sidereal day. The net effect would be that of the sun slowly moving backward — that is, from west to east — with respect to the fixed stars. If the sun were totally eclipsed during the following day also, the sun would reappear at sunrise at a point on the celestial sphere slightly behind the group of stars in which it was last seen at sunset on the previous day. During the second day the slow lagging of the sun's projection on the celestial sphere would be observed to continue. The observer would then infer that in the course of a year the projection of the sun on the celestial sphere traces a complete circle on the celestial sphere. That this circuit is a circle must be evident from the assumption that the earth's orbit lies in a plane. In fact, this path of the sun's projection on the celestial sphere must be precisely the intersection of the celestial sphere and the plane of the earth's orbit, for all the lines of projection from the moving center of the celestial sphere — the center of the earth — to the fixed sun lie in the orbital plane.

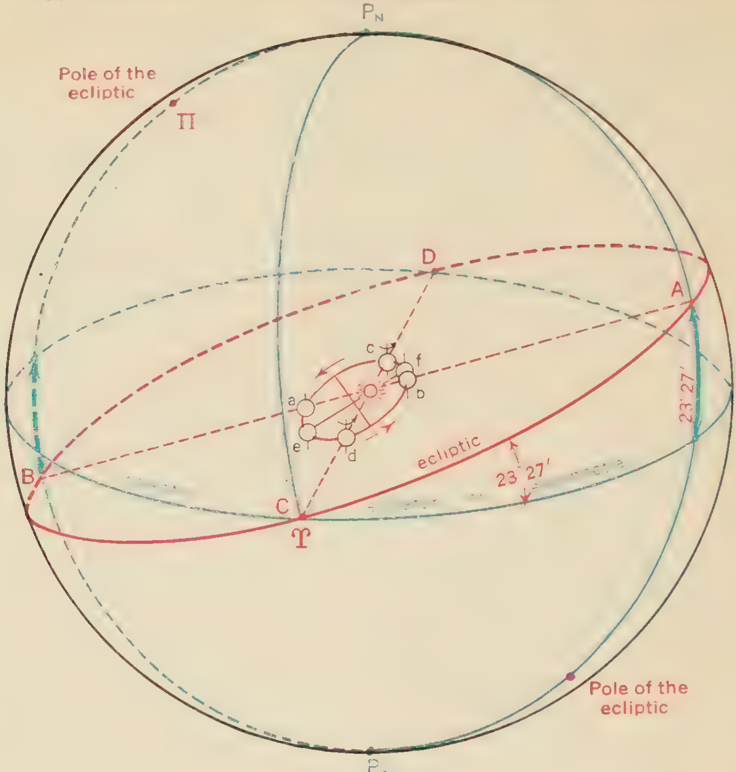


FIGURE 151

DEFINITION: *The* ***ecliptic*** *is the great-circular path of the projection of the sun onto the celestial sphere. It is therefore the intersection of this sphere and the plane of the earth's orbit.*

The position of the ecliptic on the celestial sphere can be described roughly by the particular constellations of fixed stars through which the ecliptic passes. The more important of these ecliptic constellations are, in order, Aries, Taurus, Gemini, Leo, Virgo, Scorpio, and Sagittarius. The ecliptic is shown on the star chart in Appendix IV as a dotted sine curve; the reasons for this will be explained later.

To say that at a given time "the sun is in Taurus" is to say that the sun's projection on the celestial sphere is in this constellation at this time. It would then follow that this constellation would not be visible to any terrestrial observer for a few weeks. The progress of the sun in the ecliptic can be followed throughout the year by careful observations of the celestial sphere just before sunrise and just after sunset. At these times enough stars are visible to determine in what constellation the sun is.

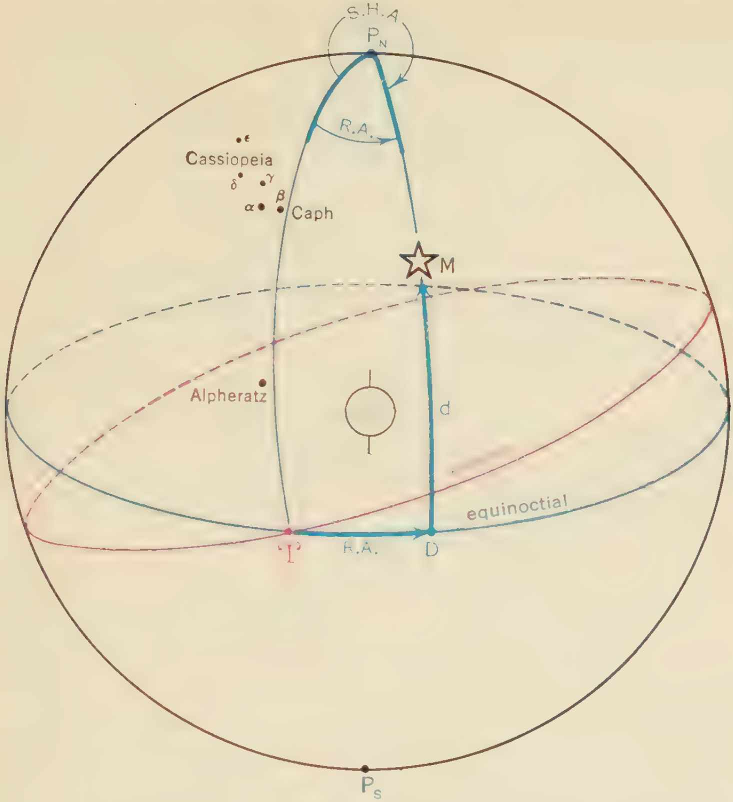


FIGURE 152

Since the terrestrial equator is inclined at an angle with the plane of the earth's orbit, it follows that the celestial equator, or equinoctial, and the ecliptic on the celestial sphere intersect at precisely this angle, the ***obliquity of the ecliptic,*** approximately equal to 23° 27′. Figure 151 pictures these two fundamental circles on the celestial sphere.

Figure 151, by showing the earth in several positions in its orbit, purports to represent the celestial sphere for all times of the year. The corresponding shifts in the center of the celestial sphere, the earth's center, are important only in so far as the projection of certain — and, necessarily, nearby — celestial bodies (the sun, for instance) are altered in *direction.* These yearly shifts in the center of the celestial sphere are immaterial as regards their *magnitude.*

Four points on the ecliptic appear of particular importance, namely, the pair of points of intersection with the equinoctial — C and D — and the pair of points halfway between the first pair — A and B. At C and D the sun is in the plane of the terrestrial equator, and hence the projections of the sun onto the celestial sphere will give points on the equinoctial, i.e., the points of intersection of the ecliptic with the equinoctial. C and D are therefore the points of projection of the sun at the "equinoxes" or the times of the year when the day equals the night. The directions of the arrows representing the senses of the axial and orbital motions indicate that c is the position of the earth at the vernal or spring equinox, about March 21, at which time the sun's projection is at C. Accordingly, this point C is appropriately labeled by the

DEFINITION*: *The fixed point on the celestial sphere which is that point of intersection of the ecliptic and equinoctial (or celestial equator) at which the sun's projection on the celestial sphere crosses the equinoctial from south to north is called* the ***vernal equinox or the first point of Aries*** *and is labeled* ♈. The vernal equinox is therefore that fixed † point on the celestial sphere into which the sun is projected from the earth's center at that instant around the twenty-first of March at which the sun is directly above the earth's equator. Points A, D, and B in Figure 151 respectively represent the ***summer solstice***,‡ for which the earth is shown at a and the date is about June 21; the ***autumnal equinox,*** for which the earth is at d and the date is about September 21; and the ***winter solstice,*** for which the earth is shown at b and the date is about December 21.

The points in the earth's orbit represented by e and f in Figure 151 are the vertices of the earth's elliptical orbit. At these times the earth is either nearest to or farthest from the sun, which is located at one of the foci of the earth's elliptical orbit.§ The earth is known to be closest to the sun (at ***perihelion***) usually early in January and farthest from the sun (at ***aphelion***) usually early in July. The orientation of the ellipse representing the earth's orbit indicates that f represents perihelion and e aphelion.

The term "fixed" as applied to ♈ demands some slight qualifications, which are of interest to the professional astronomer. The uneven pull of the planets and the moon on the earth tend to tilt slightly the plane of the earth's orbit and, therefore, the ecliptic. The uneven attraction of the sun and moon on the earth's equatorial bulge produces the phenomenon of the gyroscopic precession of the earth's axis (see the discussion of the gyro compass on page 222 in Appendix III). These factors produce a very slight motion of ♈ on the fixed celestial sphere, amounting to about 50 seconds of arc a year.

* This definition logically defines the point c in the earth's orbit in terms of point C on the celestial sphere. However, the point c is naturally more familiar than the point C.

† See the qualification of the term "fixed" below.

‡ Solstice literally means the time or point at which the sun "stands still." The sun "stands still" at these points in the sense that the north-south component of motion of the projection on the celestial sphere becomes zero at these points.

§ The earth's orbit, being an ellipse of eccentricity about $\frac{1}{60}$, is very nearly a circle.

Several hundred years ago this point on the celestial sphere was in the constellation Aries. For this reason the vernal equinox was called the "first point of Aries." Because of its slight motion this point is now slightly removed from this constellation. For our present purposes ♈ can be considered an imaginary *fixed* point on the celestial sphere — imaginary in the sense that there is nothing to mark the point.

46. Problems on Sections 43-45

1. Discuss the reasons for the following conventions in the concept of the celestial sphere:

a. Considering heavenly bodies not where they are in space but where they project to on a spherical surface.

b. Considering the center of projection not at the observer but at the earth's center.

c. Considering the radius indefinitely large instead of some arbitrarily chosen distance, in which case the heavenly bodies at a distance from the earth greater than this arbitrarily chosen radius of the celestial sphere would be thought of as being projected toward the earth's center instead of away from it.

2. Under what condition will the geocentric parallax of the moon or of a planet be zero? When is this parallax a maximum? Is the maximum parallax of the moon greater or less than the maximum parallax of a planet? Why?

3. Due to the obliquity of the plane of the ecliptic with respect to the plane of the equinoctial (i.e., the earth's axis is not normal to the plane of the earth's orbit), the earth experiences seasons which conveniently mark the passage of years. If the earth's axis were normal to the plane of the earth's orbit, what direct experience would a person on the earth have of the progress of a year?

4. In what respects would a diagram of the celestial sphere for a Martian resemble, and in what respects would it differ from, a diagram of the celestial sphere for an earth inhabitant? (The earth is 93,000,000 miles from the sun. Mars is about 141,000,000 miles from the sun and has its fixed axis inclined at an angle of 23° 30′ with the normal to its orbit which is in a plane making an angle of about 2° with the plane of the earth's orbit. The eccentricity of Mars' orbit is 0.09 and a Martian year is nearly twice as long as an earth year.)

5. If in problem 4 you replaced the word "Martian" by "an inhabitant of a 'planet' in the Andromeda Nebula," what would your answer be? See the note on page 116.

6. What is the arc distance on the ecliptic between the two equinoxes and what is the arc distance between each equinox and the solstices? What can be said of the corresponding time intervals? What calendar evidence is there that perihelion occurs between the autumnal equinox and the vernal equinox instead of vice versa? What calendar evidence is there that perihelion occurs between the winter solstice and the vernal equinox? (See page 144 (1).)

47. A Celestial Co-ordinate System Independent of the Observer: Declination and Right Ascension

Having developed the concept of the celestial sphere, we can consider our principal use of this concept: to establish celestial co-ordinate systems as a basis for numerically fixing the positions of the heavenly bodies. By means of these numerical descriptions and of certain celestial observations we can determine positions on the earth's surface.

The most fundamental system of celestial co-ordinates is one which is independent of the position of the terrestrial observer, that is, one in which the co-ordinates of the projection of each celestial body on the celestial sphere can be tabulated for use by an observer anywhere on the earth's surface.

The co-ordinate system on the earth — obviously independent of the observer — serves as a model for constructing such a system on the celestial sphere. A pair of diametrically opposite points, called "poles" of the system, are to be picked on the sphere. Great circles or "meridians" through these poles can then be considered, after which points on the celestial sphere can be partially fixed by their angular distances — latitudes — measured along the meridians from the polar of the assumed poles. Deciding upon a base meridian, from which other meridians are to be measured, will complete the fixing of points on the sphere by means of the spherical angles — longitudes — at the poles between the base meridian and the meridians of the points in question.

Convention. In the independent celestial co-ordinate system of right ascension and declination:

1. The ***poles*** are the ***celestial poles*** defined in section 44 as the projection of the terrestrial poles.
2. The ***base meridian*** is the great circle through the poles and the ***first point of Aries.***

For earth inhabitants the choice of celestial poles is obvious. Since these poles are the points towards which the earth's axis of rotation continually points, they have exceptional significance among all other points on the celestial sphere. These poles are not absolutely fixed but are fixed enough to serve.*

The choice of base meridian, however, cannot be made analogously by taking the projection of the terrestrial base meridian. The projection of the Greenwich Meridian on the celestial sphere is obviously not fixed but makes a complete circuit each sidereal day. The celestial meridian to any bright fixed star near the celestial equator could well be taken as a base meridian. Such a choice would have the advantage of being easily pictured in the sky. The first point of Aries, however, even though it cannot be seen in the sky,

* See the qualification of fixity in ♈ in the previous section.

is one of two sufficiently fixed points which possess unusual significance to earth inhabitants. The three steps indicated below will assist in obtaining a rough idea of the position on the celestial sphere of this base meridian to ♈.

1. Look for Cassiopeia and pick out the bright star, Caph (β), at the brighter end of the flat "W" in this group of stars. (See Figure 142.)

2. Imagine the meridian through Caph and note that it passes very close to another star, Alpheratz, of about the same magnitude (2.2 as against 2.4 for Caph), lying as far below Caph as Caph is below Polaris.

3. Then imagine ♈ as on this meridian so that Polaris, Caph, Alpheratz, and ♈ in this order are equally spaced from the pole to the equator. (See Figure 152.)

DEFINITION: *The spherical angle at the celestial poles between the base meridian to the first point of Aries and the celestial meridian of a celestial body is called the body's* **right ascension** *— abbreviation R.A. — when this angle is measured* eastward *from the base meridian and the body's* **sidereal hour angle** *— abbreviation S.H.A. — when this angle is measured* westward *from the base meridian.*

Right ascension has long been the standard celestial longitude co-ordinate used at sea and at astronomical observatories. The sidereal hour angle, however, is preferred in air navigation. The sidereal hour angle has the advantage of being consistent in direction with two other longitude co-ordinates measured at the celestial poles, namely, Greenwich hour angle and local hour angle, which will be discussed later. The relation between right ascension and sidereal hour angle is obviously simple, being given by

$$\text{S.H.A.} = 360° - \text{R.A.} = 24^h - \text{R.A.}$$

DEFINITION: *The* **declination** *— abbreviation* d *— of a heavenly body is its angular distance from the equinoctial, or celestial equator, measured upon the celestial meridian of that body. It is designated as north or south according as the given body is north or south of the equinoctial.*

In Figure 152, M represents a "heavenly body on the celestial sphere" — more accurately, "the projection of the body from the center onto this sphere." Its right ascension is the smaller spherical angle ♈P_NM or the arc ♈D and appears to be about 20°. The sidereal hour angle is the larger angle ♈P_NM. The declination of M is the arc DM and appears to be about 60° north. Thus the right ascension and declination, or sidereal hour angle and declination, on the celestial sphere correspond respectively to longitude and latitude on the terrestrial sphere.

Since the projections on the celestial sphere of the "fixed stars" are fixed on this sphere, we should expect the right ascension and declination of each fixed star to be constant for that particular star.

The *Nautical Almanac*, which tabulates these co-ordinates in detail for the more conspicuous stars, shows that this is very nearly the fact.*

From the above discussion of the appearances of the sun on the celestial sphere it is obvious that the right ascension of the sun is certainly not constant. It is 0° around March 21, and 90°, 180°, 270° around June 21, September 21, and December 21, respectively. Furthermore, there is a change during the year of nearly 47° in the sun's declination. It is about 23° 27′ north around June 21 and about 23° 27′ south around December 21. And yet, as previously mentioned, the sun is as fixed as all the other so-called fixed stars. This variable-co-ordinate illusion of lack of fixity of the sun is inherent in the celestial sphere itself, since this sphere was constructed by projections from the center of the earth, *which moves around the sun* but about no other star.

48. Celestial Co-ordinate Systems Dependent on the Observer: Hour Angle, Altitude, and Azimuth

If celestial observations are ever to determine the position of any given point of observation on the earth's surface, they must at some time be so described as to make them dependent on this point of observation. This can be accomplished by referring the observations to a co-ordinate system based on the observer's position. Two such systems dependent on the point of observation will be described, the first partially and the second totally dependent on the position of the point of observation.

A. The Celestial Co-ordinate System of Declination and Hour Angle

The latitude co-ordinate in this system is the same as in the absolute system, namely, declination. The longitude co-ordinate is dependent upon the position of the point of observation.

DEFINITION: *The **celestial meridian of an observer** is the projection of his terrestrial meridian onto the celestial sphere.*

* The maximum yearly change in declination of the fixed stars is about 50″. The combination of precession and nutation of the earth's axis, by slightly changing the position of the celestial poles on the fixed celestial sphere, slightly changes the position of the celestial equator, from which the declination is measured.

Both the motion of the earth's axis and tilting of the earth's orbital plane — see the second footnote on page 115 — cause slight variations in the right ascensions of the fixed stars. The changes due to the motion of the earth's axis exceed those due to the tilting of the earth's orbital plane and are more variable for the different fixed stars. The variation in right ascension of a fixed star which is due to the motion of the earth's axis depends upon the proximity of the star to a celestial pole. A slight change in the position of the celestial poles — due to the motion of the earth's axis — can easily cause a marked change in the right ascension of a fixed star extremely close to one of the celestial poles. The variation in the right ascension of Polaris, which has a declination of almost 89° north, is about 40′ of arc for the year 1943. See Appendix IV, where, in the Star Table, the S.H.A. of Polaris is qualified by being enclosed in parentheses.

The celestial meridian of an observer, therefore, rotates eastward on the celestial sphere, making one complete revolution every sidereal day.

DEFINITION: *The* ***hour angle*** *(abbreviation* t*), more specifically,* ***local hour angle*** *(abbreviation L.H.A.), of a celestial body for a particular point of observation is the spherical angle at the poles between the celestial meridian of the observer and the celestial meridian of the observed body. Hour angles are measured from the observer's meridian* ***westward*** *to the meridian of the observed body through either 360° or 24 sidereal hours.*

The celestial meridians are frequently referred to as "hour circles." In Figure 153 an observer is represented at A as having his terrestrial meridian, and therefore his celestial meridian, appear in the plane of the paper at the instant that the star M is being observed. P_NGP_S represents the meridian of Greenwich at the instant of observation, and P_NMP_S represents the hour circle of the star M. Then the angle t is the hour angle of M for the particular observer and the particular instant indicated.

Figure 153 suggests a possible later use of this hour angle co-ordinate in determining the longitude of a point of observation A. Note that t_G is the *Greenwich* hour angle of M at the instant of observation, and long. is the longitude of A. Hence, in this particular case shown (that is, for east longitude):

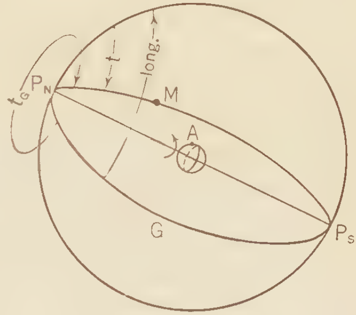


FIGURE 153

$$\text{long.} + (t_G - t) = 360°$$
$$\text{long.} = 360° - (t_G - t)$$

The Greenwich hour angle of M at any Greenwich civil time of observation, as given by a properly corrected chronometer reading, can be obtained from either the *Nautical Almanac* or the *Air Almanac*. For the latter reference see Appendix IV and note that combining the changing Greenwich hour angle of the vernal equinox, ♈, with the constant sidereal hour angle of a fixed star will give the Greenwich hour angle of the fixed star. Hence, all that is now needed to find the longitude is the hour angle, t, of M at A, the point of observation. This can never be observed but is calculated from the solution of a spherical triangle involving the co-ordinate "altitude" of M to be discussed below.

B. The Celestial Co-ordinate System of Altitude and Azimuth

The absolute celestial co-ordinates (those independent of the position of the point of observation) are tabulated in the *Nautical Almanac* and in the *Air Almanac* for various Greenwich times on various dates during the year. These tabulated absolute co-ordinates are the result of computations based on observations made with extensive apparatus at astronomical observatories. The time variable necessary for looking up any required absolute co-ordinate is given by a chronometer which indicates — with known corrections — Greenwich civil time.

Since the hour angle co-ordinate depends on the point of observation, it cannot be tabulated. Because hour angles cannot be measured by direct observation, they must be computed as unknown angles in spherical triangles in terms of a *directly observable co-ordinate* "altitude" developed below and illustrated in Figures 157 and 158.

DEFINITIONS: *The point, labeled* Z, *on the celestial sphere which, at the moment of observation, is the projection of the particular observer's position on the earth's surface is called the* ***observer's zenith*** *or the* ***zenith.*** *The diametrically opposite point on the celestial sphere is the observer's* ***nadir,*** *labeled* Na.

The zenith is therefore the point on the celestial sphere which the observer sees directly over his head at the moment. For this reason the zenith on figures of the celestial sphere is generally placed at the top of the figure. The zenith on the celestial sphere is a point continuously moving with constant angular velocity from west to east about the celestial pole and on the small circle of the celestial sphere of constant declination equal to the observer's latitude. (See Figure 154.)

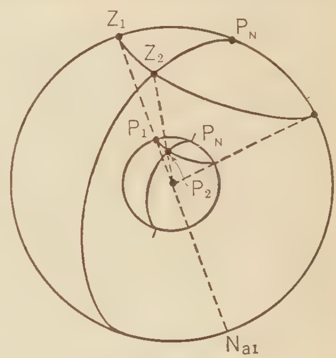


FIGURE 154

In Figure 154 P_1 is a point of observation on the earth's surface, and P_2 is a later position of this same point — due to the earth's axial rotation. Z_1 and Z_2 are the corresponding positions of the zenith of the observer, and Na_1 is the nadir of P_1.

DEFINITION: *The* ***observer's celestial horizon*** *or the* ***horizon,*** labeled NESW, *is the great circle on the celestial sphere which is polar to the zenith and the nadir.*

The *theoretical horizon* is the intersection of the celestial sphere with the horizontal plane of the observer, that is, the plane tangent to the earth's surface at the point of observation. (See Figure 155 where the scale is ignored.) The *visual horizon** is the horizon actually seen from a point neces-

* The difference between the visual horizon and the theoretical horizon is measured by the errors of *Dip* and *Refraction.* The former corrects observations based on the visual

sarily somewhat above the earth's surface, and used in making observations. These two horizons are ordinarily so closely approximated by the celestial horizon that we shall consider them as coinciding with this celestial horizon, shown in Figure 156.

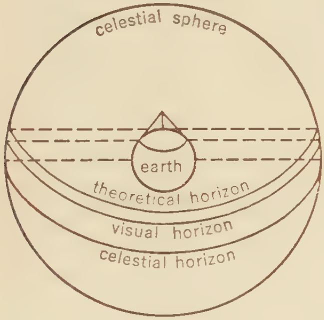


FIGURE 155

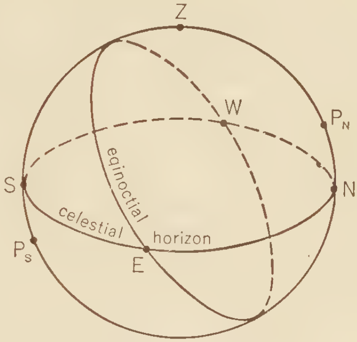


FIGURE 156

The position of a heavenly body at a given instant can now be simply described entirely in terms of the particular point of observation: The star can be said to bear from the observer so many degrees east or west of north and to be so many degrees above the horizon. This first coordinate is called the "azimuth" of the observed body and the second its "altitude." The following definitions will systematize these concepts.

DEFINITION: *The celestial great circles through the zenith are called* ***vertical circles.***

The observer's meridian is, therefore, a vertical circle, the particular one through the poles.

DEFINITION: *The intersections of the observer's meridian with the horizon are called the* ***north and the south points of the horizon.*** *They are labeled* N *and* S, *respectively.*

THEOREM: *The intersections of the horizon with the equinoctial are midway between the north and south points of the horizon and are, therefore, called the* ***east and west points of the horizon,*** *being labeled* E *and* W, *respectively.*

horizon for height of eye above the earth's surface; the latter corrects for the bending of light rays due to variations in the density of the earth's atmosphere.

The difference between the theoretical horizon and the celestial horizon is measured by the error of parallax. (See page 121.)

In Figure 156, because E and W are the intersections of the polars of Z and P_N, they are the poles of the great circle, NP_NZS, which passes through Z and P_N (see Introduction, 6 g). Hence, E and W are both 90° of arc from all points on NZP_NS and, therefore, are 90° of arc from N.

DEFINITION: *The angle at the zenith between the observer's meridian and the vertical circle of the observed celestial body is the* **azimuth** *(abbreviation* A_z*) of the body at this instant. The azimuth is measured from the part of the observer's meridian between the zenith and the nearer pole to the observed body's vertical circle and is described as being either east or west from the nearer pole.*

DEFINITION: *The angular distance on an observed body's vertical circle between the horizon and the celestial body is the body's* **altitude** *(abbreviation* h*). The altitude is described as being either above or below the horizon according as the observed body is so characterized.*

In Figure 157 the following co-ordinates for the star M are shown.

1. Its (absolute) declination, d, tabulated in the almanacs.

2. Its altitude, h, relative to the particular point of observation of latitude, lat., and longitude, long. This altitude is subject to accurate observation with a sextant at sea and a sextant or transit on land.* This altitude changes with the time of observation.

3. Its azimuth, A_z, relative to the particular point of observation. This azimuth is generally computed from other parts of the spherical triangle PZM but can be roughly observed with an azimuth circle at sea and more accurately with a transit on land.* This azimuth changes with the time of observation.

4. Its hour angle, t, relative to the particular point of observation. This hour angle is never observed. It is directly computed from tabulated data of M in an almanac when the Greenwich time and the longitude of observation are known. It can also be computed from the solution of the PZM spherical triangle when, in addition to h and d, the latitude of the point of observation is known. The hour angle changes with the time of observation.

The spherical triangle P_NZM in Figure 157 is of fundamental importance in problems involving fixing positions by means of celestial observation. It is therefore emphasized by the

DEFINITION: *The spherical triangle on the celestial sphere whose vertices are a celestial pole, an observer's zenith, and an observed star is called the* **astronomical triangle** *and is labeled by its vertices as the "*PZM *triangle."*

Examples of the solution of astronomical triangles are given in section 55.

* See Appendix III.

Figure 158 shows the star M referred to all three co-ordinate systems: the declination-right ascension system, the declination-hour angle system, and the altitude-azimuth system.

The co-ordinates of M in the three systems of co-ordinates discussed above and pictured in Figure 158 are here tabulated.

Right Ascension-Declination System	*Hour Angle-Declination System*	*Azimuth-Altitude System*
Right Ascension: $R.A. = \measuredangle \Upsilon P_N M$, measured to the east. $=$ arc ΥC	*Hour Angle:* $t = \measuredangle Z P_N M$, measured to the west.	*Azimuth:* $A_z = \measuredangle P_N Z M$
[*Sidereal Hour Angle:* $S.H.A. = \measuredangle \Upsilon P_N M$, measured to the west.]	*Declination:* $d =$ arc CM	*Altitude:* $h =$ arc BM
Declination: $d =$ arc CM		

49. Problems on Sections 47-48

Note: The excerpts from the *Air Almanac* given in Appendix IV, page 242, are essential to the solution of most of the problems below.

1. Represent each of the following stars on a separate diagram of the celestial sphere. In each case place the first point of Aries — vernal equinox — so that the particular star represented will appear on the side of the sphere out from the paper. Mark the right ascension, declination, and sidereal hour angle of each star.

(*a*) Rigel. (*b*) Alpheratz.
(*c*) Arcturus. (*d*) Kochab.
(*e*) Canopus. (*f*) Sirius.

2. In each of the following cases make a large sketch of the horizon and zenith on the celestial sphere for an observer in the given latitude. Show the poles and the equator on each sketch; likewise the *apparent* paths of the given stars, using a different color for each star. Mark each star in its apparent path at the instant for which the *first star* is directly south of the observer. Label the altitude, azimuth, and hour angle of the second star and give the value of this hour angle. Label the sidereal hour angle and declination of the *third star*.

(*a*) Lat. 30° N.; Aldebaran, Capella, Antares.
(*b*) Lat. 20° S.; Fomalhaut, Deneb, Achernar.
(*c*) Lat. 60° N.; Etamin, Kochab, Alphard.
(*d*, Lat. 50° S.; Achernar, Betelgeux, Vega.

3. For each of the following sets of data make a properly labeled sketch

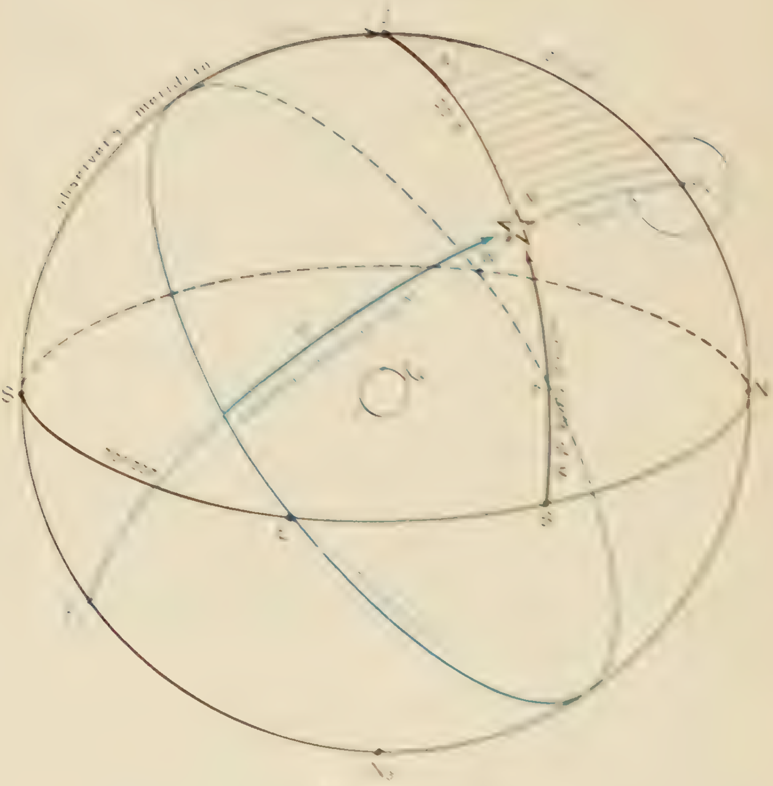


Figure 157

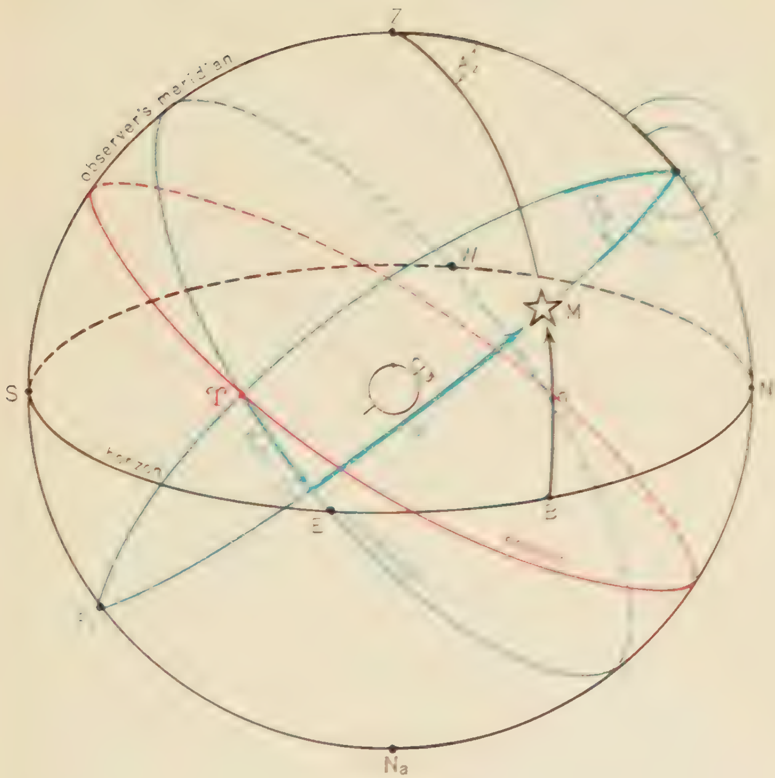


FIGURE 158

showing the horizon and zenith for an observer in the given latitude. On this sketch show the sun's *apparent* path for the given date:

(*a*) Lat. 50° N.; Winter Solstice. (*b*) Lat. 20° S.; Winter Solstice.
(*c*) Lat. 10° S.; Summer Solstice. (*d*) Lat. 40° N.; Summer Solstice.
(*e*) Lat. 0°; Autumnal Equinox. (*f*) Lat. 38° N.; Vernal Equinox.
(*g*) Lat. 70° N.; Summer Solstice. (*h*) Lat. 72° N.; Winter Solstice.

4. For an observer in the given latitude (1) which of the stars below are visible at all times on every cloudless night? (2) Which are visible at some time during some cloudless nights? (3) Which are never visible?

(*a*) Lat. 40° N.; Alioth, Canopus, Caph, Pollux, Fomalhaut.
(*b*) Lat. 25° S.; Alphecca, Kochab, Spica, Acamar, Miaplacidus.
(*c*) Lat. 5° N.; Acrux, Polaris, Dubhe, α Tri. Aust., θ Centauri.

5. In each of the following cases draw a sketch of the horizon and zenith for an observer in the given latitude. Then picture the given star when observed with the given altitude. Mark on this sketch the star's altitude, azimuth, hour angle, and declination, and also the observer's latitude. Estimate from the sketch the hour angle and azimuth of the star.

(*a*) Lat. 50° N.; Alphecca; altitude 55° in the western sky.
(*b*) Lat. 20° N.; Sirius; altitude 30° in the eastern sky.
(*c*) Lat. 30° S.; Spica; altitude 60° in the eastern sky.
(*d*) Lat. 10° N.; Mizar; altitude 30° in the western sky.
(*e*) Lat. 45° S.; Canopus; altitude 30° in the western sky.
(*f*) Lat. 10° N.; Fomalhaut; altitude 25° in the eastern sky.

6. In each of the following cases draw a sketch of the horizon and zenith for an observer in the given latitude and show the sun at the given time of day on the given date. Draw the ecliptic, and estimate from the sketch the sidereal hour angle of the projection of the observer's meridian on the celestial sphere at this instant.

(*a*) Lat. 40° N.; sunset, Vernal Equinox.
(*b*) Lat. 40° N.; sunrise, Winter Solstice.
(*c*) Lat. 10° S.; sunset, August 1, 1943.
(*d*) Lat. 0°; sunrise, August 1, 1943.

7. Assuming the visible stars to be uniformly distributed on the celestial sphere, relate the number of stars which a terrestrial observer could see during the year to the observer's latitude. What is the criterion for the possibility of an observer's seeing a given star on some cloudless night during the year?

8. (*a*) Name all the fifty-five "Navigational Stars" which are invisible in the south temperate zone (between the tropic of Capricorn, lat. 23° 27′ S., and the Antarctic Circle, lat. 66° 33′ S.).

(*b*) Name all the fifty-five "Navigational Stars" which are invisible within the Arctic Circle (lat. 66° 33′ N.).

9. For each of the following stars, state where a terrestrial observer must be if he is never to be able to see the star.

(*a*) Ruchbah. (*b*) Rigil Kentaurus.
(*c*) ϵ Argus. (*d*) Sirius.
(*e*) Peacock. (*f*) Procyon.
(*g*) Capella. (*h*) Alnilam.

10. For an observer on the equator give the time, in sidereal hours and minutes after sunset on the day of the vernal equinox and again on the day of the autumnal equinox, at which the following stars are either due north or due south that of the observer:

(*a*) Regulus. (*b*) γ Crucis.
(*c*) Betelgeux. (*d*) Hamal.

PART THREE: *Applications of Celestial Co-ordinates to Direction, Time, and Position*

50. Culminations and Elongations of Celestial Bodies

Because of the earth's axial rotation, all the stars appear to travel in small circles about the poles on the celestial sphere. For any given observer, depending only on his latitude, some of these apparent paths are totally invisible, some are partly visible, and the rest are visible in entirety. In this sense a star's apparent path is considered to be visible if its *position* on the celestial sphere is visible (that is, above the observer's horizon), whether or not, at the time, the location of the sun makes the star itself visible.

DEFINITION: *Stars whose small-circle paths are entirely visible to an observer are called* **circumpolar stars** *for the latitude of the observer.*

Stars which are circumpolar for a given latitude are always visible to an observer in this latitude on a cloudless night.

For a given observer certain points in these small circle apparent paths of the stars possess special interest. If a star is ever visible to an observer, it will at some time be visible when on the observer's meridian. In such a position the altitude of the star is instantaneously static, being midway between positions of increasing and positions of decreasing altitude. Hence, the altitude is best observed at this point. For circumpolar stars two points of crossing the observer's meridian will be visible. These points of meridian crossing are defined independently of their visibility to a given observer:

DEFINITION: *The instant at which a star crosses that half of an observer's meridian which contains the zenith [nadir] is called the instant of* ***upper* [*lower*] *culmination*** *or* ***upper* [*lower*] *transit*** *of the star for the point of observation. The star is then said to be in* ***upper* [*lower*] *culmination*** *or in* ***upper* [*lower*] *transit*** *at the point of observation.*

When a star is in culmination it is either due north or due south of the observer and is in the most favorable position for measuring altitude. In Figure 159 the upper and lower culminations of three stars are marked. One of these three stars is circumpolar for the illustrated point of ob-

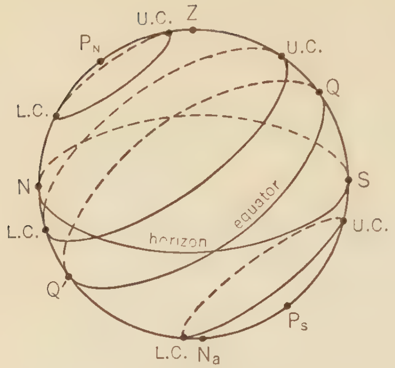


FIGURE 159

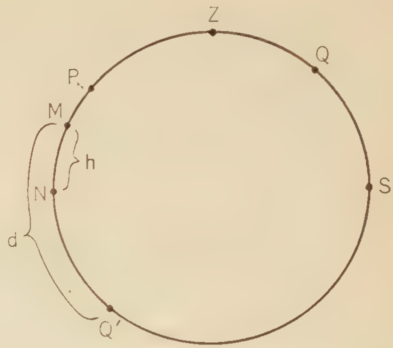


FIGURE 160

servation, and another is totally invisible for this point of observation. The instants of upper and lower culmination must occur twelve sidereal hours apart. The Greenwich times of upper culmination of certain stars at Greenwich are tabulated in the *Nautical Almanac*.

The latitude of an observer can be obtained at once by a ***meridian altitude observation,*** that is, an observation of a star in a particular culmination. The necessary calculation (see Figure 160) consists of:

a. drawing a circle to represent the observer's meridian;

b. representing thereon the observed star properly placed for its particular culmination and for the point on the horizon — north or south — from which the star's altitude was observed;

c. marking on this figure the star's tabulated declination and observed altitude; and then

d. solving for the latitude from the figure.

In Figure 160 the star M is shown in lower culmination. Q and Q' represent the intersections of the observer's meridian with the equator. Then $Q'M = d$, the star's declination, and $NM = h$, the star's altitude as measured from the north point on the horizon. Hence

$$Q'N = d - h = SQ$$
$$\text{But } QZ = \text{Latitude} = 90° - SQ.$$

The student should carefully avoid memorizing any formulas for meridian altitude problems. The fundamental concepts of declination, altitude, and upper and lower culmination, when simply illustrated on a figure, will always suffice.

For every visible star there are, as we have said, instants for most favorable observations of altitude, namely, the culminations. For

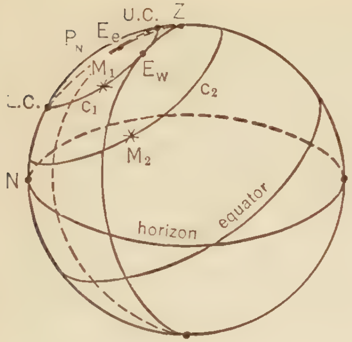


FIGURE 161

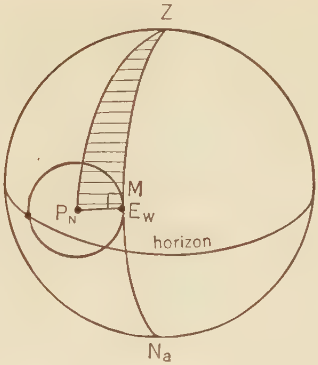


FIGURE 162

some stars there are also instants for the most favorable observations of azimuth.* Figure 161 pictures the apparent small-circle paths, c_1 and c_2, of two stars. c_1 does not but c_2 does encompass the zenith. As the star M_1 appears to move on c_1 from upper culmination, its azimuth first *increases to the west* up to a certain western maximum value, represented by the angle P_NZE_w, and then *decreases* to zero at the star's lower culmination. As the star continues in its small circle, its azimuth *increases to the east* up to a certain eastern maximum value, represented by the angle P_NZE_e, and then decreases to zero at upper culmination. The azimuth of the star M_2, on the other hand, *always decreases in western azimuth* from 180° at upper culmination to 0° at lower culmination and then *always increases in eastern azimuth* from 0° to 180° at upper culmination again. The points E_w and E_e of stars of the type of M_1 — whose small-circle paths do not include the zenith — are points at which the star's azimuth can be most accurately observed, for at such points the azimuth is momentarily static, as it is changing from an increasing function to a decreasing function, or vice versa.

DEFINITION: *When a star, which crosses the observer's meridian on the same side of the zenith at both upper and lower culminations, attains a maximum eastern [western] azimuth, the star is said to be* ***in eastern*** *[****western****]* ***elongation*** *for the point of observation.*

In Figure 161 the points of eastern and western elongation are labeled E_e and E_w, respectively.

For stars, such as Polaris, which have large declinations and hence small polar distances, the azimuth changes very slowly at the elongations. The exact times of elongations of Polaris for a given exact longi-

* Most accurately *observed* with a transit. See Appendix III.

tude and approximate latitude of observation can be computed from data in the *Nautical Almanac*. If a surveyor does not have exact time, he can watch Polaris through a transit telescope well before the estimated approximate time of elongation. The instant at which the slowly increasing azimuth begins to slowly decrease (or vice versa) will then be the instant of elongation. Having thus determined the direction from the point of observation of Polaris at elongation, the surveyor, by computing the value of this maximum azimuth, will have established a line of known true direction. By Introduction, 8 *g*, *the vertical circle and the hour circle to a point of elongation are mutually perpendicular*, since the vertical circle to a point of elongation is obviously tangent to the small-circle path of the star of this point of elongation. Consequently (see Figure 162), the computation of a star's azimuth at elongation simply involves the solution of a right *PZM* spherical triangle with the right angle at *M* in elongation. Note that, although culminations occur twelve sidereal hours apart, elongations do not, in general, occur six sidereal hours before or after a culmination. That is, the angle at P_N is not, in general, a right angle.

51. Problems on Section 50

Note: Excerpts from the *Air Almanac* and the *Nautical Almanac* in Appendix IV are to be consulted for the solutions of some of the problems in this list.

1. State the condition under which a star is circumpolar for a given point of observation on the earth's surface.

2. (*a*) Does a star which is circumpolar for a certain latitude possess points of elongation for this latitude? Illustrate with a sketch.

(*b*) Comment on the question converse to that in *a*. Illustrate.

(*c*) Under what condition is a star both circumpolar and possessed of points of elongation for a given latitude?

(*d*) Name two stars which are both circumpolar and possessed of points of elongation for points in the following latitudes: (1) 29° N., (2) 60° S.

(*e*) Using Introduction, 8 *f*, on a sketch approximately locate the points of elongation in the apparent path of the star Vega for an observer in latitude 20° N.

3. What can be said of the time intervals between eastern and western elongations of a given star and the observer's latitude?

4. Find the latitude of the points of observation for which the following tabulated meridian-altitude data apply.

Observed Altitude of the Star	*Bearing of the Star*	*Type of Culmination*	*Tabulated Declination of the Star*
(*a*) 40°	North	Lower	70° north
(*b*) 10° 30′	North	Upper	30° 20′ north
(*c*) 52° 25′	South	Upper	06° 35′ north
(*d*) 26° 18′	South	Lower	41° 13′ south

5. What can be said of the number of circumpolar stars relative to the observer's position on the earth's surface? What is the numerical condition for a given star to be circumpolar for a given observer?

6. On May 30, 1943 (see Polaris Table in Appendix IV), in latitude 35° 37′ 30″ N., a surveyor, having estimated the approximate hour of the approaching western elongation of Polaris, continuously observes this star with his transit. At the instant of elongation, as determined by the star's apparent vertical motion, the surveyor clamps together the upper and lower plates of the transit to prevent horizontal motion of the telescope and then has a stake set in the line of sight of the telescope. Through what angle must the telescope then be turned to set a second stake due north of the point over which the transit is placed?

7. Solve problem 6 for the following data and find also the hour angle of Polaris at elongation.

Date	*Latitude*	*Kind of Elongation*
(*a*) July 19, 1943	16° 05′ 40″ N.	eastern
(*b*) Dec. 15, 1943	62° 14′ 15″ N.	western
(*c*) April 15, 1943	49° 25′ 40″ N.	western
(*d*) Jan. 4, 1944	38° 05′ 30″ N.	eastern

8. Find the latitude of the points of observation for which the following tabulated meridian-altitude data apply:

Star	*Bearing of Star*	*Type of Culmination*	*Observed Altitude*
(*a*) Canopus	South	Lower	21° 49′ 20″
(*b*) Achernar	South	Upper	61° 10′ 40″
(*c*) Bellatrix	North	Upper	32° 42′ 30″
(*d*) Denebola	South	Upper	53° 18′ 00″
(*e*) Alioth	North	Lower	24° 50′ 40″

52. Time and the Mean Sun

Reference has repeatedly been made to the time at which certain observations are made and in particular to the Greenwich time of various celestial phenomena. It is now necessary to systematize the general concept of time and to define the various systems in current use.

Since a solar day is the time interval between two successive coincidences of the projection of any given terrestrial meridian with the projection of the sun on the celestial sphere, and since (because of the earth's orbital revolution) the sun's projection is not fixed but slowly moves "backward" — that is, from west to east — along the ecliptic by about one degree a day, the length of a solar day depends not alone on the earth's axial rotation but also upon its orbital revolution. The axial rotation of the earth is extraordinarily constant, which means that sidereal days are of constant length. But the fact that the length of days depends on the orbital revolution of the earth makes for two difficulties in securing the prime requisite for timekeeping — days of equal length. The difficulties are these:

1. The sun's projection on the ecliptic does not move with a constant speed.

This speed exactly reflects (as it is caused by) the earth's orbital revolution. The earth and the sun, like all other pairs of bodies in space, are attracted to one another by a force which increases as the distance separating them decreases. The earth's orbit is an ellipse with the sun at one focus. Since the earth's speed of revolution in its orbit is a measure of the force of attraction between the earth and the sun, the speed of orbital revolution of the earth must be greatest at perihelion and least at aphelion.

2. The sun's projection moves on the ecliptic and not on the equinoctial. Consequently, even if the backward — that is, west to east — motion of the sun's projection on the ecliptic were uniform, the corresponding backward turning of the sun's celestial meridians at noon on successive days would not be uniform.

In Figure 163 the sun's projection is shown at two different points, A_1 and A_2, on the ecliptic, that is, for two different times of the year. The celestial meridians of A_1 and A_2 are pictured as intersecting the equinoctial at C_1 and C_2, respectively. P_NQ is the meridian to the point on the equinoctial 90° of arc from the first point of Aries. Then, if A_1 and A_2 are on opposite sides of P_NQ, by Napier's Corollary 3

♈ $C_1 <$ ♈ A_1 and ♈ $C_2 >$ ♈ A_2.

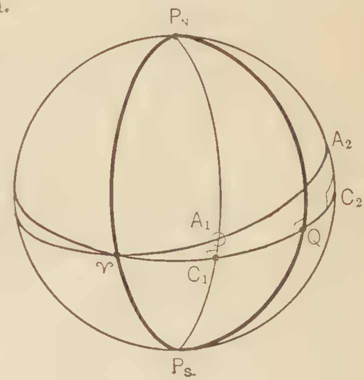


FIGURE 163

Hence, if A_1 moves uniformly along the ecliptic from the first point of Aries, C_1 will not move uniformly along the equinoctial but will alternately move more slowly and more rapidly.

But this backward motion of the sun's meridians represents the daily amount by which the length of a solar day is greater than the length of a sidereal day. Since sidereal days are of constant length, solar days, if measured between successive coincidences of the projection of a given terrestrial meridian with the meridian of the sun's projection, will not be constant. (Since the earth's orbit is only slightly elliptical, the eccentricity being $\frac{1}{60}$, this second of the two causes of solar days of varying length is the more important.)

Because days of varying length would be impossible to measure by any but impractically intricate clocks and watches, the sun's actual projection on the ecliptic is replaced, for purposes of mechanically measuring time, by the Mean Sun.

DEFINITION: *The **mean sun** is a fictitious heavenly body having the three properties:*

1. *It moves in the equinoctial (not in the ecliptic).*

2. *Its speed in the equinoctial is constant, and, since it completes one circuit of the equinoctial in one year, its speed is the mean speed of the sun in the ecliptic.*

3. *Its position is always near the projection of the real sun by virtue of the fact that the right ascension of the mean sun is made equal to the mean **celestial longitude** of the sun, where by*

> DEFINITION: ***Celestial longitude** is longitude on the celestial sphere measured by meridians through the pole of the ecliptic, the base meridian being through the first point of Aries.*

The difference between the mean celestial longitude and the actual celestial longitude of the sun is specified to be zero at perihelion.

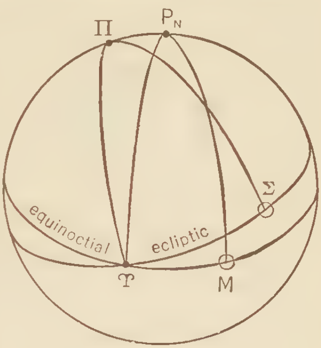


FIGURE 164

> In Figure 164 Σ is the projection of the true sun at a given time of the year, and M is the position of the mean sun at this same time. Π is the pole of the ecliptic. Then angle ♈ΠΣ is the celestial longitude of the true sun, and angle ♈ P_NM is the right ascension of the mean sun. Angle ♈P_NM equals angle ♈ΠΣ at perihelion. At other times angle ♈P_NM is equal to the celestial longitude of the sun *assuming* its projection on the ecliptic moved with *constant* speed beginning at perihelion.

Time based on this mean sun can be recorded by watches and clocks, since the days in such a system of time will be of equal length.

There are four distinctly different systems of time in current use. Each system is frequently called by alternate names to emphasize the feature of the system which is particularly pertinent to the discussion at hand. These four systems of time are listed below with their alternate titles. The definitions describing each system follow this listing:

I. *Mean time;* synonymous with

- *a.* mean solar time, local mean solar time,
- *b.* civil time, or local civil time.

II. *Apparent solar time;* synonymous with

- *a.* true solar time,
- *b.* local apparent time.

III. *Sidereal time;* synonymous with
  a. local sidereal time.

IV. *Zone time;* synonymous with *
  a. standard time,
  b. "clock time," legal time (modified zone time in some cases),
  c. summer time, daylight saving time, war time (modified legal time).

I. Mean time
  a. is based on the mean sun,
  b. measured from midnights,
  c. uniformly elapsing,
  d. not directly observed but determined by correcting the directly observed apparent time,
  e. recorded by chronometers.

DEFINITION: *A* ***mean solar day*** *is the time between two successive lower culminations of the mean sun with any given terrestrial meridian — more exactly, with the projection of this terrestrial meridian on the celestial sphere.*

DEFINITIONS: *The instant of* upper [lower] *culmination of the mean sun with any given meridian is called* ***mean noon****, or* ***local mean noon*** [***mean midnight, or local mean midnight***], *at this meridian for this particular day.*

Mean solar days are therefore measured from the midnight of the *previous* day to the midnight of that day.

Previous to 1925 each mean solar day began at noon. The term "mean civil time" for mean time emphasizes the fact that since 1925 mean solar days begin at midnight as do civil days.

Each mean day is divided into twenty-four equal hours beginning at 0^h at midnight of the previous day and running to 24^h at midnight of that day.

DEFINITION: *The* ***mean time*** *at any point on the earth's surface at any given instant is the hour angle of the mean sun at this point increased by twelve hours.*†

DEFINITION: ***Greenwich civil time*** (*or, less explicitly,* ***Greenwich time***) *is mean time at Greenwich, England.*

Chronometers are kept at all observatories and on all ships to indicate Greenwich time. (See Appendix III, Part II.)

II. Apparent solar time
  a. is based on the true sun,

* Notice the modifications. † Modulo 24.

b. measured from midnights,
c. not uniformly elapsing,
d. directly observed,
e. not directly recorded but corrected from the chronometer.

DEFINITIONS: ***Apparent solar day, apparent noon, apparent midnight,** and **apparent time** are defined exactly as are the corresponding terms in the mean time system by replacing the term "mean sun" by the term "true sun."*

DEFINITION: *The **equation of time** is the difference between apparent time and mean time. At any given instant the equation of time consists of a signed solar time interval in minutes and seconds to be applied to the mean time at this instant to give the apparent time at this instant.*

Since the mean sun, unlike the true sun, cannot be directly observed, chronometers for recording the uniformly elapsing mean time are checked by first directly observing the nonuniformly elapsing apparent time and then correcting this by the equation of time. The equation of time can be read from either almanac; directly from the *Nautical Almanac* and by means of simple calculations from the *Air Almanac.* The maximum numerical value of the equation of time is between sixteen and seventeen minutes of time (see problem 11, section 53).

III. Sidereal time
a. is based on the first point of Aries (the vernal equinox),
b. measured from noons,
c. uniformly elapsing,
d. directly observed,
e. recorded by sidereal clocks.

DEFINITION: *A **sidereal day** is the time between two successive* upper *culminations of the first point of Aries (the vernal equinox).*

DEFINITION: *The **sidereal time** at any point on the earth's surface at a given instant is the hour angle of the first point of Aries at this point.*

Since hour angles are measured from the observer's meridian *westward* to the celestial body's meridian, and since right ascension is measured from the meridian of the first point of Aries *eastward* to a given meridian, it follows (see Figure 165) that *the sidereal time along any terrestrial meridian at a given instant is the right as-*

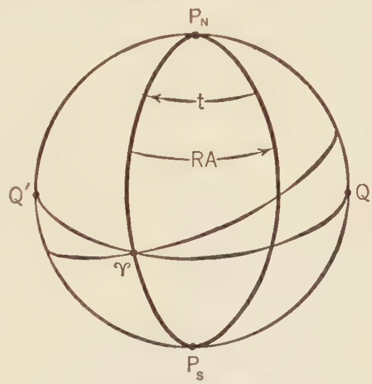


FIGURE 165

cension of this terrestrial meridian — or its projection on the celestial sphere — at this instant. Sidereal days are divided into 24 equal hours beginning at 0^h when the first point of Aries is in *upper* transit at the given terrestrial point ("sidereal noon") and running up to 24^h when this situation next obtains. Clocks or watches adjusted to keep sidereal time are called ***sidereal clocks*** or ***sidereal watches.*** They will necessarily gain very nearly four minutes a day on ordinary clocks and watches regulated to keep mean solar time. The truth of the following theorem is immediate:

THEOREM: *For a given observer the hour angle of a fixed star at any instant is the time in sidereal units since the fixed star was last "on the observer's meridian"; that is, since the projection of the observer's meridian last coincided with the star's hour circle.*

IV. Zone time

- *a.* is based on mean time at meridians whose longitudes are integral multiples of 15°,
- *b.* measured from midnights,
- *c.* uniformly elapsing,
- *d.* not directly observed but checked by reference to chronometers,
- *e.* recorded by watches and clocks.

DEFINITION: ***Zone time*** *is the mean solar time at the nearest meridian whose longitude is an integral multiple of* 15°.

These meridians, therefore, define the centers of the respective time zones. The zone clock time at a given instant is, therefore, the same for every point in a given zone, and the zone times at a given instant at any two points on the earth's surface differ, if at all, by an integral number of hours. Figure 166 illustrates the division of the earth's surface into time zones. Note that the zone for which the 180° meridian is the center is peculiar in that, though the *watch* time throughout this whole zone is everywhere the same, the *actual* time in the western half of this zone is exactly one day later than the time in the eastern half of this zone.

DEFINITION: *The* ***legal time*** *of any locality on the earth's surface is the zone time of either the time zone in which this locality lies, or the nearer neighboring time zone when statutory regulations so decree for the sake of avoiding confusion in the midst of populous areas.*

For this reason the International Date Line does not exactly follow the 180° meridian but varies from it to avoid passing through large masses of land. Watches and clocks in common daily use are set to the legal times of the localities in which they are in use.

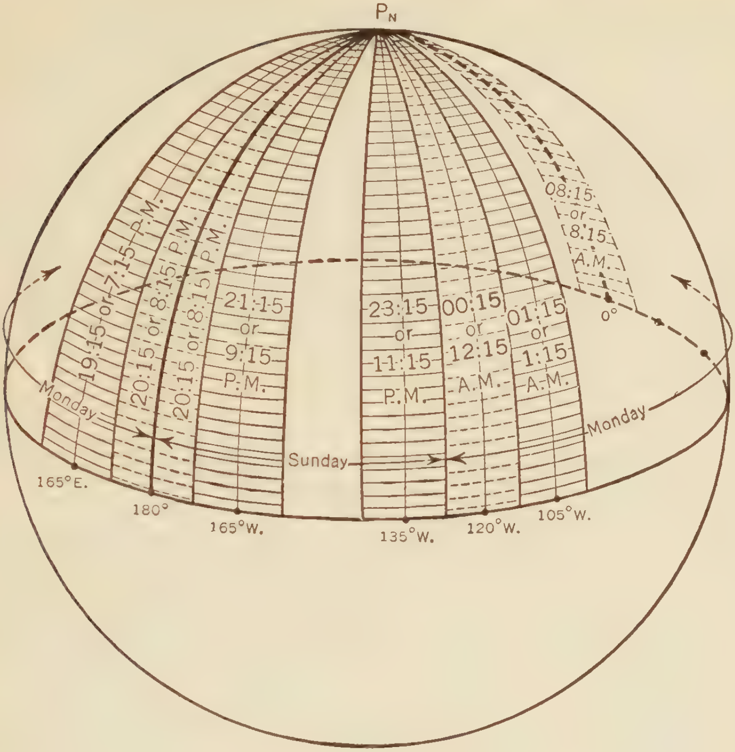


FIGURE 166

DEFINITION: ***Summer time, daylight-saving time,*** *or* ***war time*** *are temporary modifications of the above defined legal time (and then themselves become legal time) by which the time in any given locality is decreed to be one hour later than the regularly defined legal time.*

In civil practice the twenty-four hour legal day is customarily divided into two series of twelve hours each, one series, labeled A.M., running from midnight of the previous day to noon and the other series, labeled P.M., running from noon to midnight of that day.

Summary of Time Systems

Sidereal time is theoretically the most satisfactory system of time. It is directly observable by means of instants of transits of stars and it is uniformly elapsing. But, because the sun is a so much more spectacular body than any other star, solar time is the more natural system for daily use. Because apparent solar time is not uniformly elapsing, it is modified by the theoretical concept of the mean sun to give the uniformly elapsing, but not directly observable, mean time. Finally, zone time and legal time are practical adaptations of mean time to the earth's surface as a whole; in zone time and legal time noons will occur in each locality within less than an hour of the instants at which the sun is highest in the heavens for that locality (except that in the case of the temporary time systems this local difference between legal and true noon may be as much as nearly two hours), and at any given instant the minute hands of all ordinary time pieces will everywhere indicate the same number of minutes.

53. Problems on Section 52

Note: The excerpts from the *Air Almanac* given in Appendix IV are essential to the solutions of many of the problems in this list.

1. By means of a suitable figure prove the

THEOREM: *The sidereal time of* 0^h *mean solar time at any point on the earth's surface is equal to twelve hours plus the right ascension of the mean sun at that instant.*

State the relation between the sidereal time and the sun's sidereal hour angle.

2. (*a*) On a figure of the celestial sphere show the equinoctial, ecliptic, mean sun, true sun, and the projection of the Greenwich meridian at 0^h Greenwich civil time on August 1, 1943. (See the *Air Almanac* for this day.)

(*b*) Find the equation of time at 0^h G.C.T. on August 1, 1943, and again at 1:42:26 P.M., G.C.T. (Suggestion: Compare the G.H.A. of the true and mean sun.)

3. (*a*) On a figure of the celestial sphere show the equinoctial, ecliptic, vernal equinox, mean sun, true sun, and the projection of the Greenwich meridian at 5:20 P.M., G.C.T. on August 1, 1943.

(*b*) Find the sidereal time at the beginning of this day at Greenwich and again at noon. By how much has sidereal time gained or lost on mean solar time in this half day? (Note that 1^h of time = 15° of arc; 1^m of time = 15′ of arc, etc.)

4. Draw a large sketch of the celestial sphere showing the horizon and zenith for an observer in the given latitude and also the equinoctial and ecliptic for the given sidereal time. On this sketch show the sun on the ecliptic at the given hour of local apparent time. From this sketch estimate the approximate time of year:

(*a*) Lat. 30° N., sidereal time 3^h, local apparent time 2 P.M.
(*b*) Lat. 60° N., sidereal time 20^h, local apparent time 4 A.M.
(*c*) Lat. 20° S., sidereal time 4^h, local apparent time 11 P.M.

5. Find the local zone time, that is, watch time without war time or daylight time, and the local mean time for the following apparent times:

(*a*) 9:08:24 A.M. at New York (lat. 40° 40′ N., long. 73° 50′ W.) on August 1, 1943.

(*b*) 2:04:08 P.M. at Moscow (lat. 55° 45′ N., long. 37° 36′ E.) on August 1, 1943.

(*c*) 7:32:48 A.M. at Tokyo (lat. 35° 40′ N., long. 139° 45′ E.) on August 2, 1943.

(*d*) 4:12:20 P.M. at Honolulu (lat. 21° 18′ N., long. 157° 51′ W.) on July 31, 1943.

6. Find to within 4 seconds the San Francisco watch time, that is, war time for the particular time zone, at which it will be apparent noon in San Francisco (lat. 37° 45′ N., long. 122° 27′ W.) on August 1, 1943. What will be the local mean time at this instant?

7. Find the sidereal time at the following places for the given zone war times.

(*a*) $21^h\,20^m$ at Dallas, Tex. (lat. 32° 47′ N., long. 96° 48′ W.) on July 31, 1943.

(*b*) $03^h\,40^m$ at Cairo, Egypt (lat. 30° 02′ N., long. 31° 21′ E.) on August 1, 1943.

(*c*) $05^h\,10^m$ at Dunedin, N.Z. (lat. 45° 52′ S., long. 170° 32′ E.) on August 2, 1943.

8. Find the right ascension, declination, and local hour angle of the planet Venus at Cape Town (lat. 33° 55′ S., long. 18° 22′ E.) at 9:40 P.M. local watch time — assuming war time — on August 1, 1943.

9. In each of the following cases a terrestrial observer on August 1, 1943, measures the altitude of the given heavenly body at upper culmination and notes the Greenwich civil time of this observation. Find the position of the observer in each case.

Star or Planet	*Altitude at Upper Culmination*	*Direction of Star or Planet*	*G.C.T. of Observation*
(*a*) Rigil Kent.	65° 08′ 30″	South	$16^h\,41^m\,23^s$.
(*b*) Antares	49° 30′ 00″	South	$08^h\,19^m\,14^s$.
(*c*) Mars	65° 05′ 00″	North	$02^h\,07^m\,18^s$.
(*d*) Venus	24° 36′ 20″	South	$21^h\,22^m\,43^s$.
(*e*) Acamar	40° 53′ 00″	South	$12^h\,13^m\,03^s$.
(*f*) Ruchbah	65° 08′ 40″	North	$07^h\,03^m\,55^s$.

10. On August 1, 1943, at the given known longitudes the upper transits of the following stars were observed at the given observed chronometer instants. Compute the error in the chronometers in indicating G.C.T.

(*a*) long. 116° 41′ W., Deneb, $08^h\,05^m\,10^s$.

(*b*) long. 29° 54′ E., Altair, $20^h\,53^m\,05^s$.

11. *The Figure-Eight-Shaped Diagram of the Equation of Time Frequently Seen on Globes of the Earth.*

(*a*) Assuming that the earth's orbital motion were uniform (that is, considering just the second factor discussed in section 52 as necessitating the convention of the mean sun), how many times during a year would you expect the equation of time to change sign? One of the instants at which the equation of time is zero in 1943 occurs on Christmas Day, which is just after the winter solstice, December 22, and just before perihelion, very early in January. Accordingly, under the assumption given above, give the approximate period or periods in the year during which the equation of time would be respectively positive and negative.

(*b*) Assuming the earth's orbit were in the plane of the equinoctial (that is, considering just the first factor discussed in section 52 as necessitating the convention of the mean sun), how many times during a year would you expect the equation of time to change sign? Give the approximate period or periods in the year during which the equation of time would be respectively positive and negative.

(*c*) The earth's orbit, being elliptical with eccentricity $\frac{1}{60}$, is almost circular, so that the earth's orbital motion is very nearly uniform. Consequently, the fact that the true and mean suns are on different circles on the celestial sphere is much more significant than the earth's variable orbital motion in the equation of time. Accordingly, how many times during the year would you expect the equation of time actually to change sign?

(*d*) For the year 1943 the extreme and zero values of the equation of time occur sometime during the following days:

Feb. 12	$-14^m\,20^s.9$	July 27	$-\ 6^m\,23^s.0$
Apr. 16	$00^m\,00^s.0$	Sept. 2	$00^m\,00^s.0$
May 14	$+\ 3^m\,46^s.1$	Nov. 3	$+16^m\,21^s.9$
June 14	$00^m\,00^s.0$	Dec. 25	$00^m\,00^s.0$

Explain in the light of parts (*a*), (*b*), and (*c*) why the magnitudes of the extreme values of the equation of time and likewise the time intervals between these extreme values and the next zero values are so variable. Note that the equation of time is roughly pictured on artificial globes of the earth by a diagram in the shape of a figure eight, usually placed in the eastern Pacific Ocean. One loop of this "eight" is much longer and wider than the other loop because of the facts explained above. (See Figure 167.)

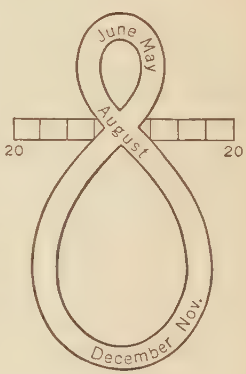


FIGURE 167

54. Altitude Observations for Determining Position: Lines of Position

As the earth rotates at the center of the celestial sphere of fixed stars, different points on the earth's surface are successively brought directly underneath any particular star.

DEFINITION: *A point on the earth's surface is at a given instant the* ***substellar*** [***subsolar***] ***point*** *of a given star* [*the sun*], *if the particular star* [*the sun*] *is at that instant at the zenith of the point.*

The substellar or subsolar point is the point on the earth's surface at which the star's or sun's altitude is instantaneously 90°. Since celestial declination, which is practically constant for any given fixed star, is measured from the projection of the terrestrial equator, from which terrestrial latitude is measured, it is evident (see Figure 168) that those points on the earth's surface which at some time during the day are sub-

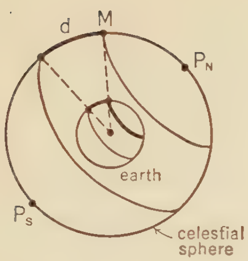


FIGURE 168

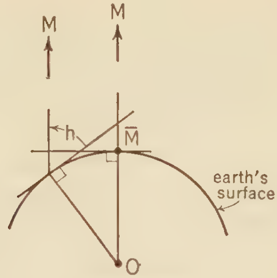


FIGURE 169

stellar with respect to that star lie approximately on the same parallel of latitude and on that parallel for which the latitude is equal to the star's declination.

THEOREM: *A point on the earth's surface at which a given star*, M, *has an altitude* h *must at the instant lie on that small circle of the earth whose pole*, $\overline{\text{M}}$, *is the substellar point of* M *and whose arc distance from* $\overline{\text{M}}$ *is equal to the complement of* h.

This follows directly from Figure 169 when it is recalled that lines of sights to a fixed star from two different points on the earth's surface must be considered parallel, because of the great remoteness of fixed stars from the earth.

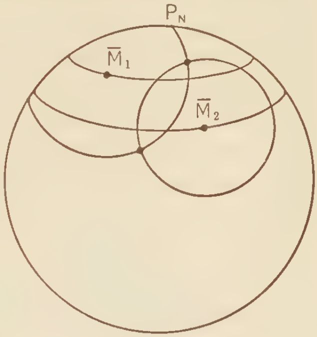


FIGURE 170

This theorem gives a theoretical means for determining the hitherto totally unknown position of a point of observation: If altitudes of two different stars, M_1 and M_2, are observed at the same time, two small circles are determined on each of which the point of observation must lie. The poles of these small circles are the substellar points, $\overline{M}_1$ and $\overline{M}_2$, of M_1 and M_2, respectively (see Figure 170). The latitudes of $\overline{M}_1$ and $\overline{M}_2$ are equal to the declinations of M_1 and M_2, respectively. The longitudes of $\overline{M}_1$ and $\overline{M}_2$ are essentially the Greenwich hour angles of the stars at the instant of observation.*

The Greenwich hour angle of any heavenly body at any time is readily obtainable from either the *Nautical Almanac* or the *Air Almanac*. The

* If the Greenwich hour angle of the star is less than 12^h or 180°, the longitude of the substellar point is this hour angle *west*. If the Greenwich hour angle of the star is greater than 12^h or 180°, the longitude of the substellar point is 24^h or 360° minus this hour angle *east*.

excerpts from the latter in Appendix IV are sufficient to provide this information at any time on August 1, 1943. Note that in the case of a fixed star the Greenwich hour angle is found by combining the variable Greenwich hour angle of the vernal equinox, ♈, with the constant sidereal hour angle of the star.

Finally, the polar distances, in nautical miles, of the two small circles are equal to the observed co-altitudes, in minutes, of the corresponding stars.

Unfortunately, the determination of the latitudes and longitudes of the points of intersection of these two small circles about $\overline{M}_1$ and $\overline{M}_2$ (one of these points of intersection being the point of observation) is neither direct nor simple. Since no chart exists for which the scale of distances from any given point is the same in all directions, these small circles cannot be drawn on a chart. On a Mercator chart the correct intersection of these two circles could be approximated by the additional observation of the azimuths of the stars M_1 and M_2.* Then, using the scales proper for these approximate azimuth directions from $\overline{M}_1$ and $\overline{M}_2$, the navigator could scale off the co-altitude distances in these general directions to give an approximate point of intersection as the point of observation. But, except in the case of very high altitudes — which are to be avoided as being difficult to obtain accurately — the co-altitude distances to be scaled off from $\overline{M}_1$ and $\overline{M}_2$ would be so large that any chart large enough for sufficiently accurate graphical results would be impractically cumbersome.† One of the special tabulation methods of navigation provides very special graphs from which the positions of the points of intersection of these small circles can be read.‡ Such special tables are particularly useful in air navigation where the need for rapidity of calculation frequently warrants a slight sacrifice in accuracy. Otherwise, that is, universally at sea and to some extent in the air, the method described above is modified to give the line of position method described below.

When the complete absence of any information as to a navigator's position is replaced by a reasonably accurately *estimated* position, this estimated position can be *corrected* to give the *actual* position to a high degree of accuracy by a modification of the more complicated method described above, which is necessary when no previous information is available concerning the position of the point of observation. At sea the record of courses steered and speeds maintained (corrected for wind

* At sea this is obtainable within half a degree or so by means of an azimuth circle (see Appendix III). In the air this is not practical.

† Since observed altitudes are generally between 20° and 70°, the corresponding distances to be scaled off from the substellar points would lie between 4200 and 1200 nautical miles.

‡ *Star Altitude Curves* by Lieutenant Commander P. V. H. Weems.

and current) from the previous point of definitely known position is sufficient to give an approximate position known as a ***dead reckoning position.*** Correcting this dead reckoning position to give the actual position involves the solution of an astronomical spherical triangle, on the basis of which a simple graphical procedure will give the actual position on a Mercator chart.

Any program for finding the actual position from an assumed position by means of altitude observations would naturally suggest investigating the differences in a star's altitudes from two different points of observation. The following corollary to the above theorem is therefore pertinent:

COROLLARY 1: *The difference between the nautical mile distances of two points of observation from the instantaneous substellar point of a given star is equal to the difference, in minutes of arc, between the instantaneously observed altitudes of this star at these two points, the point at which the altitude is greater being the nearer to the substellar point.*

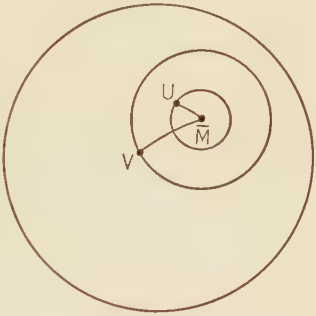


FIGURE 171

Applying the above theorem to Figure 171 where U and V are any two points of observation of the star M whose substellar point is instantaneously at $\overline{M}$, and where h_U and h_V are the altitudes of M as observed at U and V, respectively, we have

$$\overline{M}V \text{ (in nautical miles)} = (90° - h_V) \text{ in minutes.}$$
$$\overline{M}U \text{ (in nautical miles)} = (90° - h_U) \text{ in minutes.}$$
$$\overline{M}V - \overline{M}U \text{ (in nautical miles)} = (h_U - h_V) \text{ in minutes.}$$

The following is an obvious specialization of the above corollary:

COROLLARY 2: *The distance in nautical miles between two points on the same great circle through the instantaneous substellar point of a star is equal to the difference in minutes between the instantaneous altitudes of the star as observed from the two points, the point of the larger altitude being the nearer to the substellar point.*

The application of Corollary 1 is seen to be immediate in the following procedure for finding the actual position as a correction on a dead reckoning position:

1. The altitude of a certain star is observed with a sextant and the chronometer time of observation is noted. This *observed* altitude is labeled h_o and the corresponding point of observation, that is, the actual position, is labeled A_o.

2. For this chronometer time of observation the dead reckoning posi-

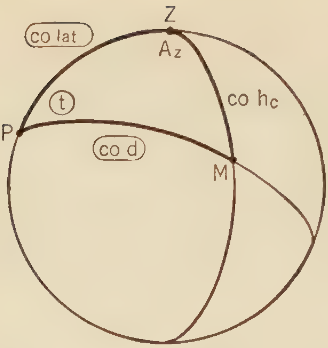


FIGURE 172

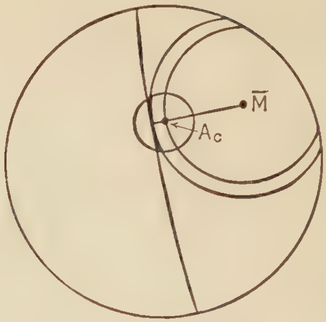


FIGURE 173

tion is calculated from the record of the ship's run from the last definitely known position. This information and the star's declination — tabulated in an almanac — provide three parts of an astronomical spherical triangle. In Figure 172 the known parts of such a triangle are shown encircled. This triangle is then solved for h_c, the *computed* altitude of the star, which altitude therefore applies to the dead reckoning position, labeled A_c. The azimuth, A_z, of the star is also computed for this point A_c.

3. The points A_o and A_c can now replace the points U and V in Corollary 1 with the known restriction that A_o lies inside a relatively very small "small" circle about A_c. (See Figure 173.) Without loss of generality in the argument A_c is pictured as nearer $\bar{M}$ than is A_o. The scale is greatly exaggerated to make the small circle about A_c distinct, since A_c is assumed to be within about 20 miles of A_o, whereas the distances of A_o and A_c from $\bar{M}$ are, in general, between 1200 and 4200 miles. Consequently, it is wholly reasonable to replace the small-circle arc which is within the small circle about A_c and on which A_o must lie by the great-circle arc which is tangent to this small-circle arc at its midpoint.

4. Accordingly, following the solution of the astronomical spherical triangle, the procedure on a Mercator chart * is as follows:

a. The dead reckoning position of A_c is plotted. (See Figure 174.)

b. Through A_c a line is drawn in the direction of $\bar{M}$ as given by the computed azimuth of M.

c. At B, on this line and at a scale distance from A_c equal (in nautical miles) to the numerical value (in minutes) of the difference $(h_o - h_c)$, away from or toward $\bar{M}$ according as h_o is less or greater than

* A plane chart on which lines of constant direction on a sphere are represented as straight lines and on which angles are preserved.

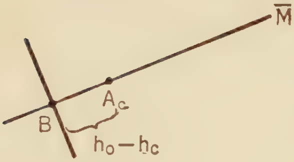


FIGURE 174

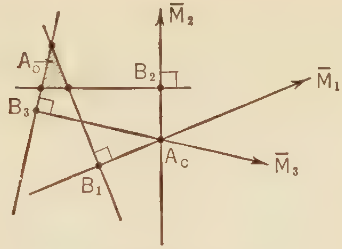


FIGURE 175

h_c, a perpendicular line is drawn on which the point A_o, of the actual position, must lie. This perpendicular line is accordingly called a ***line of position*** of the point of observation.

5. By repeating the above procedure for another star of markedly different azimuth the point A_o will be determined by the intersection of two lines of position. In practice this procedure is generally followed for three stars M_1, M_2, and M_3 so that A_o is "fixed" by being shown to lie in a small triangle formed by three lines of position (see the shaded triangle in Figure 175).

The practicability of this line of position method of determining position as a correction on a dead reckoning position lies in the relatively short distances involved. Whereas the substellar point of an observed star is generally from one to four thousand miles away, the dead reckoning position is probably, in the cases in which this method is justified, less than twenty miles away.* The fact that the simple Mercator chart procedure of this line of position method must be preceded by the solution of at least two astronomical spherical triangles makes this method a direct application of spherical trigonometry.

55. Solutions of Celestial Problems

The procedure to be followed in solving a celestial problem is as follows:

1. Draw a large sketch of the celestial sphere. If the data are given in terms of a particular observer, place his zenith at the top of the meridian in the plane of the paper. If the observer's latitude is given, show the poles and the equinoctial to suggest the value of this latitude. Either the east or the west horizon can be faced outward according to the side of the zenith on which the elevated pole — that corresponding to the observer's latitude — is placed. If possible, the choice should be made

* When the assumed position has been reckoned from a definitely known position of several days past, this method may well give the actual position as 50 to 100 miles away from the assumed position. In such cases the method can be reapplied by using in place of the originally assumed position the position just computed.

so that any heavenly body mentioned in the problem will be shown outward from the paper.

2. Show on this celestial sphere any celestial bodies mentioned and in the approximate position given. Show any co-ordinates which are given or which are obtained from an almanac.

3. Resolve the problem into the solution of some spherical triangle or triangles. This will frequently involve a fundamental *PZM* celestial triangle. See, for example, the triangle P_NZM in Figure 176. Redraw this spherical triangle (or triangles) disconnected from the sphere, drawing a circle around the labels for the known parts.

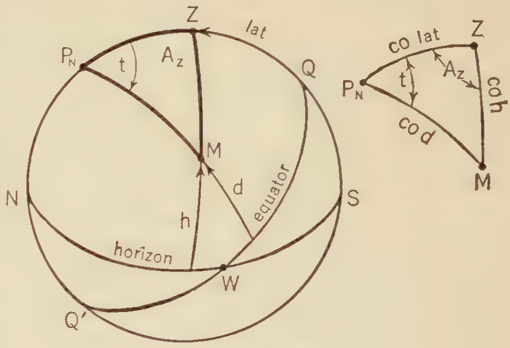


FIGURE 176

4. Solve this spherical triangle, or these spherical triangles, by the method of reduction to right spherical triangles discussed in Chapters 2 and 3. Reduce the answers to the form required.

EXAMPLE 15: On a certain day for which the *Nautical Almanac* gives the sun's declination as 15° 27′ 24″ South, a navigator observes the altitude of the sun with his sextant as the sun nears the navigator's meridian. He notes that at the instant the sun's altitude ceases increasing and begins to decrease, the chronometer indicates $09^h\ 53^m\ 20^s$ G.C.T., the greatest altitude attained (at this instant) being 65° 34′ 40″ above the northern horizon. What is the latitude and longitude of the navigator? (See Figure 177.)

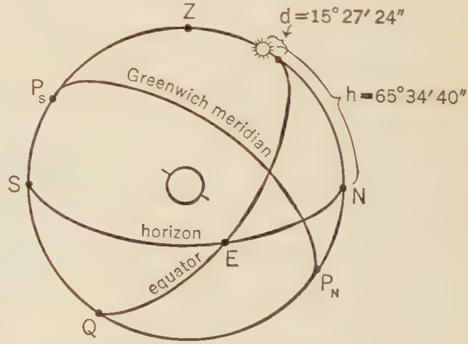


FIGURE 177

Since the sun is south of the equator and is observed north of the observer's zenith, the observer must be in the southern hemisphere.

From Figure 177, $h - d =$ co lat.

$$\therefore \quad \text{lat.} = 90 - h + d = \boxed{39°\ 52'\ 44''\ \text{S.}}$$

$$\text{long.} = 11^h 59^m 60^s - 09^h 53^m 20^s = \boxed{31°\ 40'\ \text{E.}}$$

EXAMPLE 16: The latitude of Annapolis, Maryland, is 38° 59′ N. On a particular sunny winter day when the declination of the sun (as given by

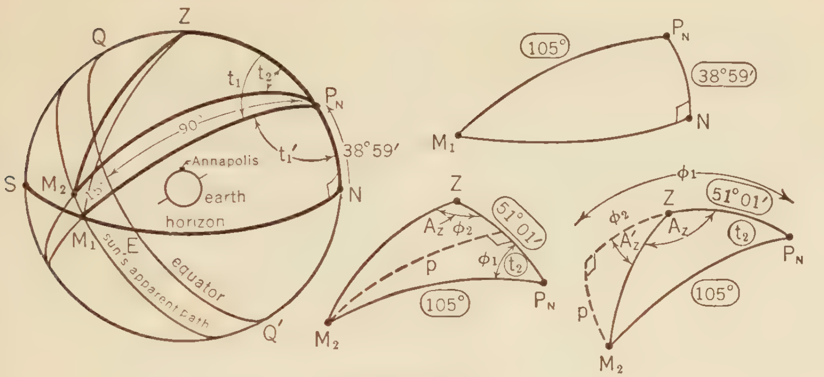


FIGURE 178

the *Nautical Almanac*) is 15° South, the sun appears to rise directly at the foot of East Street, and one hour later the telegraph poles on Main Street cast shadows parallel with the curb. In what directions do these two streets run from State Circle and Church Circle, respectively? (See Figure 178.)

$\cos M_1N = \sec 38°\,59' \cos 105°$ $\qquad\qquad$ $\cos 105° = \cos M_1N \cos 38°\,59'$

$\cos t_1' = \tan 38°\,59' \cot 105°$

$t_1 = 11^h\,59^m\,60^s - t_1'$ (or $179°\,59'\,60'' - t_1'$)

$t_2 = t_1 - 1^h\,00^m\,00^s$ (or $t_1 - 15°\,00'\,00''$)

$\sin p = \sin 105° \sin t_2$ (same quadrant as t_2)

$\tan \phi_1 = \tan 105° \cos t_2$ $\qquad\qquad$ $\cos t_2 = \cot 105° \tan \phi_1$

~~$\phi_2 \neq 51°\,01' \neq \phi_1$~~ $\qquad$ $\phi_2 = \phi_1 - 51°\,01'$

$\cot A_z^{(\prime)} = \sin \phi_2 \cot p$ $\qquad\qquad$ $\sin \phi_2 = \tan p \cot A_z$

38° 59′	l sec 10.10940	l tan 9.90811		
105°	l cos (−) 9.41300	l cot(−) 9.42805	l sin 9.98494	l tan (−) 10.57195
$M_1N = 109°26'56''$	l cos (−) 9.52240			
		l cos (−) 9.33616		

$t_1' = \begin{cases} 102°\,31'\,26'' \\ 6^h\,50^m\,06^s \end{cases}$

$t_1 = \begin{cases} 5^h\,09^m\,54^s \\ 77°\,28'\,34'' \end{cases}$

$t_2 = \begin{cases} 4^h\,09^m\,54^s \\ 62°\,28'\,34'' \end{cases}$ $\qquad$ l sin 9.94784 $\quad$ l cos 9.66475

$p = 58°\,56'\,16''$ $\qquad$ l cot 9.77984 $\qquad$ l sin 9.93278

$\phi_1 = 120°\,06'\,22''$ $\qquad\qquad$ l tan (−) 10.23670

$\phi_2 = 69°\,05'\,22''$ $\qquad$ l sin 9.97041

$A_z' = 60°\,38'\,06''$ $\qquad$ l cot 9.75025

∴ (*a*) East Street runs 19° 26′ 56″ South of East
(*b*) Main Street runs 29° 21′ 54″ South of East.

Example 17: On August 1, 1943, soon after sunset when the horizon and brighter stars were visible, a navigator, by means of a sextant, observed the altitude of the star Altair. When duly corrected for height of eye and instrument error, this observed altitude was found to be 35° 15′ 40″. The Greenwich time (G.C.T.) of this observation — obtained from the watch time of the instant of observation, the difference between watch time and chronometer time, and the chronometer error — was computed to be $22^h\ 33^m\ 18^s$. By dead reckoning the ship's position was assumed to be lat. 52° 30′ N., long. 31° 15′ W. Compute the direction and distance of the ship's line of position from this assumed dead reckoning position and illustrate graphically.

The following abbreviations will be used:

- G.C.T. Greenwich civil time or Greenwich time.
- G.H.A. Greenwich hour angle.
- S.H.A. Sidereal hour angle.
- L.H.A. Local hour angle.
- A.A.A. *American Air Almanac*. See references to Appendix IV.
- lat. Latitude.
- long. Longitude.
- d Declination
- A_z. Azimuth.
- t Time angle. This equals either the hour angle or 360° minus the hour angle, whichever is the smaller.
- h_c Computed altitude.
- h_o Observed altitude.
- ♈ The vernal equinox or the first point of Aries.

Given: Date: August 1, 1943. G.C.T. = $22^h\ 33^m\ 18^s$.
lat. = 52° 30′ N. Star: Altair.
long. = 31° 15′ W. h_o = 35° 15′ 40″.

Reduction of data to give three parts of an astronomical triangle:

1.	G.H.A. of ♈ at $22^h\ 30^m$	= 287° 09′	(See A.A.A. Aug. 1, 1943.)
2.	Correction for $03^m\ 18^s$	= 00° 50′	(See A.A.A., Interpolation for G.H.A.)
3.	G.H.A. of ♈ at $22^h\ 30^m\ 18^s$	= 287° 59′	
4.	S.H.A. of Altair	= 63° 00′	(See A.A.A., Stars.)
5.	G.H.A. of Altair	= 350° 59′	
6.	long. (W.)	= 31° 15′	
7.	L.H.A. of Altair	= 319° 44′	
8.	t for star in eastern sky	= **40° 16′**	
9.	d of Altair	= **08° 43′ N.**	(See A.A.A. Stars.)
10.	lat.	= **52° 30′ N.**	

And 8, 9, and 10 are either known parts or known complements of parts of the astronomical triangle to be solved for h_c and A_z. (See Figure 179.)

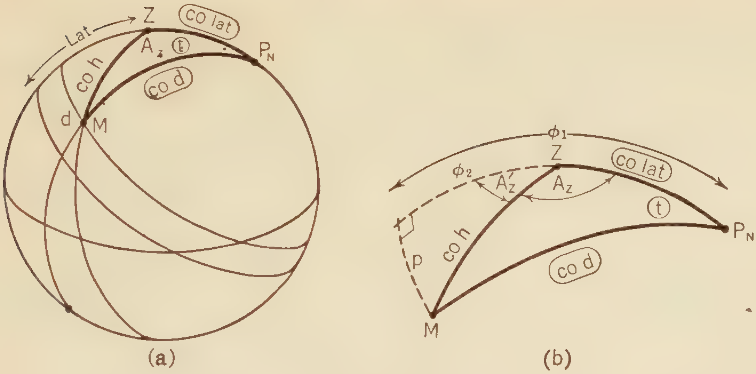


FIGURE 179

Solution of the Astronomical Triangle:

Figure 179 indicates that the triangle is of the s.a.s. type. Since the unknown angle at M is not required, the altitude is drawn from this vertex.

$\sin p = \sin t \cos d$

$\tan \phi_1 = \cos t \cot d$ $\qquad\qquad$ $\cos t = \cot (\text{co } d) \tan \phi_1$

~~$\phi_2 = \text{co lat.} - \phi_1$~~; $\phi_2 = \phi_1 - \text{co lat.}$

$\sin h_c = \cos p \cos \phi_2$

$\cot A_z' = \cot p \sin \phi_2$ $\qquad\qquad$ $\sin \phi_2 = \tan p \cot A_z$

t	= 40° 16′ 00″	l sin 9.81047	l cos 9.88255		
d	= 08° 43′ 00″	l cos 9.99495	l cot 10.81440		
p	= 39° 42′ 32″	l sin 9.80542		l cos 9.88609	l cot 10.08067
ϕ_1	= 78° 38′ 20″		l tan 10.69695		
lat.	= 52° 30′ 00″				
co lat.	= 37° 30′ 00″				
ϕ_2	= 41° 08′ 20″			l cos 9.87686	l sin 9.81815
h_c	= 35° 24′ 20″			l sin 9.76295	
A_z'	= 51° 36′ 54″				l cot 9.89882
h_o	= 35° 15′ 40″				
$h_c - h_o$	= 00° 08′ 40″				
	= $8\frac{2}{3}$ n. mi.				

Therefore the ship's line of position is $8\frac{2}{3}$ *nautical miles from the assumed position, the* $8\frac{2}{3}$ *nautical miles being measured on a line bearing 51° 36′ 54″ west of north, that is,* away *from the star with respect to the assumed position.* (*See Figure* 180.)

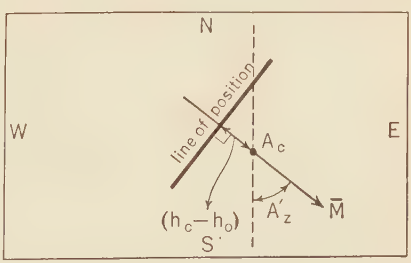


FIGURE 180

56. Problems on Chapter 5

Note: The excerpts from the almanacs given in Appendix IV are to be used in many of the problems in this list.

1. For the following data find the unknown quantities:

	Altitude (h)	*Horizon*	*Elevated Pole*	*Declination* (d)	*Latitude* (*lat.*)	*Hour Angle* (*H.A. or t*)	*Azimuth* (A_z)
(*a*)	29°18′30″	East		14°10′20″ S.	28°10′30″ N.		
(*b*)	32°08′30″		North	52°08′10″ N.		$20^h27^m18^s$	
(*c*)	61°10′15″	West	North	08°15′22″ S.			N. 142°25′ W.
(*d*)	41°50′00″	West		07°42′40″ N.	19°05′20″ S.		
(*e*)				20°32′40″ N.	29°50′40″ S.	$3^h04^m10^s$	
(*f*)				19°50′20″ S.	32°42′30″ N.	$3^h05^m23^s$	
(*g*)	65°03′30″		South	57°08′40″ S.		$22^h08^m20^s$	
(*h*)				06°42′10″ S.	48°15′40″ S.	$20^h12^m08^s$	

2. Find the direction and distance of the line of position from the assumed dead reckoning position for each of the following sets of data in which the Greenwich civil time of observation is that for August 1, 1943. Illustrate your answer graphically.

	Assumed latitude (*lat.*)	*Assumed Longitude* (*long.*)	*Aug. 1, 1943 G.C.T.* (*G.C.T.*)	*Star Observed*	*Observed Altitude* (h_o)
(*a*)	12° 45′ S.	152° 30′ W.	$16^h\ 08^m\ 19^s$	Fomalhaut	36° 27′ 20″
(*b*)	22° 30′ N.	134° 45′ E.	$20^h\ 02^m\ 08^s$	Markab	51° 55′ 20″
(*c*)	18° 30′ S.	61° 30′ E.	$13^h\ 33^m\ 22^s$	Antares	58° 55′ 30″
(*d*)	54° 30′ N.	28° 45′ W.	$22^h\ 12^m\ 47^s$	Ruchbah	33° 05′ 30″
(*e*)	09° 00′ N.	42° 24′ W.	$21^h\ 35^m\ 12^s$	Nunki	28° 15′ 00″
(*f*)	32° 30′ S.	78° 58′ W.	$11^h\ 40^m\ 14^s$	Aldebaran	36° 45′ 20″
(*g*)	35° 45′ N.	18° 30′ E.	$03^h\ 32^m\ 49^s$	Rigel	20° 28′ 00″
(*h*)	53° 30′ S.	63° 14′ W.	$11^h\ 21^m\ 40^s$	Acrux	33° 14′ 20″

3. Find the zone time — without war time or daylight time — in each of the following places on August 1, 1943, at the instant when a vertical post casts a shadow in the given direction. (*Note:* If your first answer shows that the sun's assumed declination was in error, make the necessary changes in your solution.)

(*a*) Brunswick, Me. (lat. 43° 55′ N., long. 69° 59′ W.); 72° 30′ east of north.

(*b*) Brockport, N.Y. (lat. 43° 13′ N., long. 77° 57′ W.); 58° 15′ west of north.

(*c*) Babylon, L.I. (lat. 40° 40′ N., long. 73° 20′ W.); 37° 30′ east of north.

4. In each of the following cases find, by means of a solution of a spherical triangle, the local apparent time of sunrise and the direction of the shadow cast by a vertical post at sunrise.

(Note that the motor-vehicle laws of many states set the times of turning on and off headlights as one-half hour after sunset and one-half hour before sunrise, respectively.)

Place	*Latitude*	*Sun's Declination*
(*a*) Toledo, Ohio	41° 40′ N.	05° 14′ 10″ N.
(*b*) Toronto, Canada	43° 38′ N.	18° 52′ 45″ S.
(*c*) Fairbanks, Alaska	64° 44′ N.	20° 20′ 00″ N.
(*d*) Mexico City, Mexico	19° 00′ N.	22° 00′ 20″ N.
(*e*) Buenos Aires, Argentina	34° 35′ S.	21° 52′ 10″ S.
(*f*) Auckland, New Zealand	36° 52′ S.	12° 22′ 40″ N.
(*g*) Quito, Ecuador	00° 10′ S.	00° 10′ 00″ S.
(*h*) Hammerfest, Norway	70° 38′ N.	17° 24′ 20″ S.
(*i*) Murmansk, Russia	68° 50′ N.	18° 14′ 30″ S.
(*j*) Manchester, England	53° 30′ N.	22° 10′ 15″ N.

5. In each case of problem 4 find the direction of the shadow cast by a vertical post at 2:15 P.M. local apparent time.

6. In each of the following cases a navigator assumes his latitude and observes the altitude of a star at the given Greenwich civil time of August 1, 1943, as computed from a chronometer, in order to compute his longitude. Find the navigator's longitude. (*Note:* This method is justifiable even when the latitude is somewhat in doubt, if the star whose altitude is observed is nearly due east or due west. See the Method of Lines of Position, however, for the more modern practice.)

Latitude of Observation	*Star Observed*	*Observed Altitude*	*Horizon Used*	*Aug. 1, 1943 G.C.T.*
(*a*) 38° 27′ 30″ N.	Vega	53° 10′ 30″	Eastern	$21^h\ 31^m\ 10^s$
(*b*) 15° 05′ 00″ S.	Enif	40° 08′ 40″	Western	$17^h\ 08^m\ 12^s$
(*c*) 31° 53′ 15″ N.	Alnilam	23° 00′ 30″	Eastern	$13^h\ 56^m\ 38^s$

7. A ship in latitude L (north) sails continuously on a course 270° at k knots. Let R be the radius of the earth in nautical miles and let the (northern) declination of the sun be d (assumed constant for the day). Show that the number of hours, T, of sunlight which the ship experiences on this day is given by

$$T = \frac{2\,\pi R\,(180° - t')\cos L}{15\,\pi R\cos L - 180\,k}, \quad \text{where } \cos t' = \tan L \tan d$$

8. Derive an analogous expression for T in the above problem, if the course is 90° and the declination, d, is southern.

9. On March 11, 1943, a surveyor in latitude 38° 11′ 45″ N., long. 110° 05′ 20″ W. wishes to locate the meridian by sighting on Polaris at elongation, setting a stake in this direction, and then turning his transit through a computed angle from the line of this stake to true north.

(*a*) At what zone time should the surveyor be prepared to sight on Polaris? Will the elongation be eastern or western?

(*b*) Using the latitude of the point of observation and the declination of Polaris, find the direction of true north with respect to the direction of the first stake from the transit. (See *Nautical Almanac* table for Polaris in Appendix IV.)

10. Solve problem 9 for the following sets of data:

Date	*Latitude*	*Longitude*
(*a*) Nov. 5, 1943	22° 17′ 20″ N.	82° 15′ 30″ E.
(*b*) Aug. 8, 1943	34° 24′ 45″ N.	63° 48′ 10″ W.
(*c*) Dec. 1, 1943	58° 19′ 20″ N.	133° 24′ 15″ W.

11. In each of the following cases a navigator assumes his longitude and observes the altitude of a star at a given Greenwich civil time of August 1, 1943, as computed from a chronometer, in order to compute his latitude. Find the navigator's latitude. (*Note:* This method is justifiable even when the longitude is somewhat in doubt, if the star whose altitude is observed is near the observer's meridian. The method of lines of position, in which both latitude and longitude are assumed, is more commonly used now.)

	Longitude of Observation	*Star Observed*	*Observed Altitude*	*Elevated Pole*	*Aug. 1, 1943 G.C.T.*
(*a*)	15° 03′ 00″ W.	Mizar	22° 14′ 00″	North	$04^h\ 31^m\ 13^s$
(*b*)	29° 52′ 30″ E.	Peacock	23° 06′ 30″	South	$04^h\ 42^m\ 13^s$
(*c*)	174° 43′ 00″ W.	Vega	66° 03′ 30″	North	$08^h\ 05^m\ 25^s$

12. Prove that for any locality not on the equator the local apparent time of sunrise on the day of an equinox is halfway between the times of sunrise at the solstices. What is the situation for localities on the equator?

13. What is the latitude of a locality for which the earliest local apparent time of sunrise is $2\frac{1}{2}$ hours earlier than the time of latest sunrise? Use 23° 27′ as the maximum numerical value of sun's declination.

14. By means of a large figure of the celestial sphere show that for any point "inside" the Arctic Circle (that is, north of this circle) there will be days on which the sun will not set. (The co-latitude of the Arctic Circle is equal to the angle between the ecliptic and the equinoctial, or 23° 27′.)

15. Boston, Mass. (lat. 42° 15′ N., long. 71° 00′ W.), New York, N.Y. (lat. 40° 40′ N., long. 73° 50′ W.), and Charleston, S.C. (lat. 32° 47′ N., long. 79° 57′ W.) all have the same legal time. Assume this legal time is zone time (not war time nor daylight time). State the time order of occurrence of (1) apparent noon, (2) mean solar noon, and (3) legal or clock time noon at each of these three cities on August 1, 1943.

16. A surveyor on land in lat. 09° 20′ 12″ N.; long. 08° 37′ 14″ W. wishes to lay off a direction bearing 155° 18′ 10″ west of north from a certain point on the night of August 1, 1943. Because local hills obscure Polaris, he proposes to do this by setting up his transit at this point and taking a sight on the star Antares at the instant when this star bears in this desired direction. At what Greenwich time and at what zone time should he make this observation?

17. At what mean solar time in San Francisco (lat. 37° 45′ N.; long. 122° 27′ W.) is the sidereal time on August 1, 1943, $05^h\ 37^m\ 40^s$?

18. Find the sun's right ascension, declination, and local hour angle on August 1, 1943, at 10:20 A.M. zone time in New York (lat. 40° 40′ N.; long. 73° 50′ W.).

19. If on the afternoon of August 1, 1943, a surveyor in lat. 32° 47′ N. and in the sixth time zone west of Greenwich measures the altitude of the sun with a transit and finds it to be 51° 42′, through what horizontal angle must he rotate the transit telescope to set a stake on the meridian due south?

20. By means of a solution of a spherical triangle compute the L.A.T. (local apparent time), the L.M.T. (local mean time), and the W.T. (watch time) of sunrise at the following places on August 1, 1943. Compare your results with the times of sunrise as tabulated in the *Air Almanac* for certain latitudes on

this day. (*Note:* Use for the sun's declination a value at a reasonable hour of G.C.T. If the declination at the computed hour is different from this, correct your computations accordingly.)

(*a*) Reykjavik, Ice. (lat. 64° 04′ N., long. 21° 58′ W.).
(*b*) Cape Horn (lat. 55° 59′ S., long. 67° 16′ W.).
(*c*) Rochester, N.Y. (lat. 43° 08′ N., long. 77° 35′ W.).

21. Find the terrestrial latitude and longitude of the substellar point of each of the following stars at the given instant of G.C.T. on August 1, 1943. By means of a map relate these points to some large city:

(*a*) Deneb; $06^h\ 17^m\ 35^s$.
(*b*) Markab; $18^h\ 24^m\ 02^s$.
(*c*) Kaus Australis; $01^h\ 39^m\ 04^s$.

22. What large city is very nearly the substellar point of Dschubba on August 1, 1943, at $22^h\ 11^m\ 46^s$ G.C.T.? If the altitude of Dschubba, bearing northwest, was observed to be 82° at this instant, approximately where was the point of observation?

23. Compute the zone time of upper transit of the following stars at the given places on August 1, 1943:

(*a*) Antares; Manila (lat. 14° 35′ N., long. 121° 00′ E.).
(*b*) Alpheratz; Washington, D.C. (lat. 38° 55′ N., long. 77° 00′ E.).

24. (*a*) A casual observer in northern latitude is puzzled to note that the sun continues to rise later by his watch for several days after the winter solstice. Explain how this can be.

(*b*) Using the data below (taken from the *Nautical Almanac*) compute, by means of solutions of spherical triangles, the watch times of sunrise at Chicago, Ill. (lat. 41° 50′ N., long. 87° 40′ W.) on December 22, 1943, the winter solstice, and on December 31, 1943.

	Dec. 22, 1943	*Dec. 31, 1943*
Sun's mean declination................	23° 26′ .7 S.	23° 08′ .7 S.
Equation of Time for 0^h G.C.T.........	$+ 1^m\ 56^s.6$	$- 2^m\ 31^s.2$
Hourly difference in Equation of Time..	$- 1^s.2$	$- 1^s.2$

(*Note:* Equation of Time = Apparent Time − Mean Time.)

25. *Sundials*

(*a*) Show why a sundial cannot, in general, be made from a thin vertical stick mounted on a horizontal board. What celestial co-ordinate of the sun does the shadow of such a stick indicate? What co-ordinate of the sun should a shadow indicate to constitute a sundial? Are there points on the earth's surface at which a vertical stick on a horizontal board will constitute a sundial? Discuss the shadows cast by a vertical stick on a horizontal board at the equator on a day of an equinox.

(*b*) Show that a sundial can be constructed if the thin stick, called the "style" or "gnomon," which is mounted perpendicularly to the board (or "face" of the sundial), is pointed toward the elevated celestial pole. (See Figure 181.) Note that, although the center of the celestial sphere (that is, the earth's center) moves by nearly 200 million miles a year, this displacement is infinitesimal in comparison with the infinite radius of the celestial sphere. Hence, any displacement of the center of the celestial sphere to place it at some point of observation on the earth's surface will be indeed

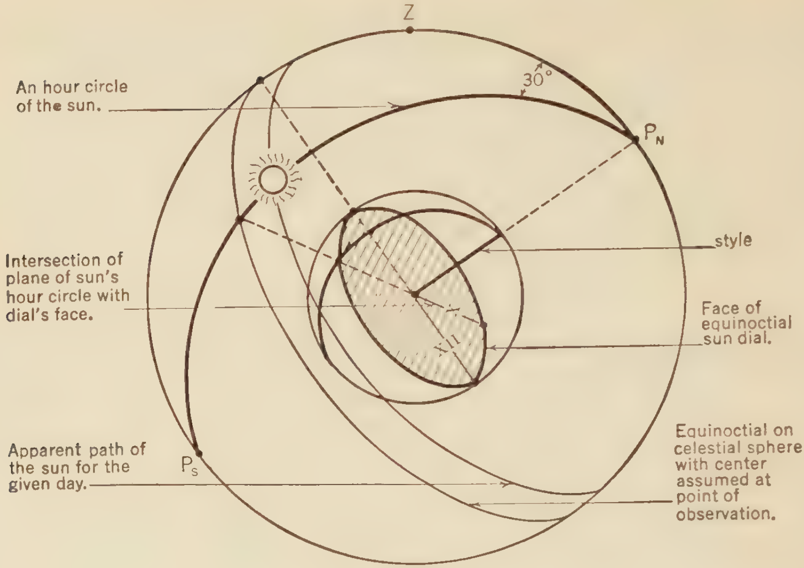


FIGURE 181

inconsequential. This type of sundial is called an ***equinoctial sundial,*** as its face is in the plane of the equinoctial when its style is correctly pointed to the elevated pole. What kind of time will be indicated by this sundial? Describe precisely how the rays which are on the face of the dial and emanate from the foot of the style must be drawn. If an equinoctial sundial is correctly set up, that is, with the style parallel to the earth's axis, will it be usable at any point on the earth's surface?

(*c*) Most sundials have either horizontal faces or vertical faces which face toward a cardinal point of the compass. The former are called ***horizontal sundials*** and the latter ***vertical sundials.*** To construct a horizontal sundial, let a plane intersect the face of an equinoctial sundial along the east-west line of this face and at an angle with the style equal to the latitude of the wabe in question. (See Figure 182.) This plane will then be horizontal and will be the face of a horizontal sundial. In terms of the latitude of the wabe and by means of Figure 182 derive the formula for the angle θ_n on the face of a horizontal sundial between the line marking apparent noon and the line marking n hours before or after apparent noon. What can be said about the limitations of a horizontal sundial with respect to its usability at different points on the earth's surface? Compare the horizontal sundial with the equinoctial dial in this respect.

(*d*) By means of a figure similar to Figure 182 derive the formula for marking off the hour lines on a vertical sundial facing south for northern latitudes or facing north for southern latitudes. To what pole will the style of such vertical sundials point? Comment on the year-around usability of such a vertical sundial placed on the wall of a building. Compare such a vertical sundial with a horizontal sundial in this respect.

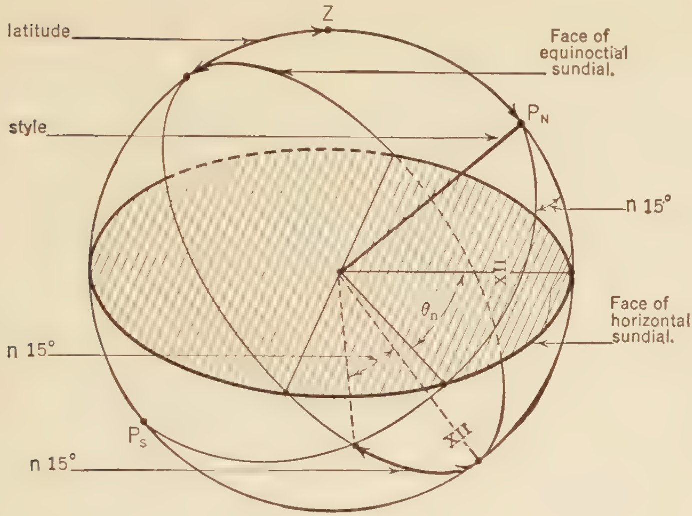


FIGURE 182

(*e*) Imagine that a window in an Oxford College building looks out upon a vertical outside wall across a court. Suppose this vertical wall faces 20° west of south, and suppose a vertical sundial is to be placed on this wall so that the time of day can be read from an opposite window. Compute the angle from the vertical on the face of this dial at which the five o'clock (P.M.) shadow line must be drawn. The latitude of Oxford is 51° 45′ N.

Appendixes

Geometrical Description and Classification of Ambiguous Solutions

1. The Program

In the text proper, only right ambiguous spherical triangles have been described geometrically. (See section 16.) In the case of oblique ambiguous spherical triangles we were content to do three things: first, to recognize whether or not a given triangle to be solved was ambiguous *; second, to know in general how a constructed altitude was to be placed with respect to the possible double solutions †; and third, to admit no solution, one solution, or two solutions for the particular problem at hand entirely on the basis of the numerical computations.‡ In what follows here the geometrical description of ambiguous solutions will be extended to oblique spherical triangles. This is a much more difficult program than the geometrical description of the right ambiguous triangles. Several general theorems will first be developed. As a result of the geometrical description of these ambiguous oblique spherical triangles, the types of solution possible in each of the two general cases will be classified on the basis of the values of one of the three given parts relative to the values of another given part and the computed value of the altitude needed for the solution of the particular triangle at hand.

2. Ambiguous Right Spherical Triangles: Geometrical Description and Classification of Solutions.

For the sake of completeness the results of section 16 are listed here without explanation:

DEFINITION: ***Ambiguous right spherical triangles*** *are those for which the data, beside the assumed right angle, comprise a leg and opposite angle.* (See sections 16 and 16 *a*.)

* The given parts include a pair of opposite parts.

† The altitude must lie inside one and outside the other triangle in the a.s.s. case and either inside both or outside both triangles in the s.a.a. case.

‡ Log sine and log cosine must not be positive; log secant and log cosecant must not be negative; parts of triangles must be positive and less than 180°.

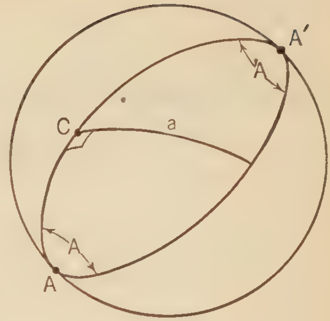


Figure 183

Geometrical Description: In the case of double solutions the two solution triangles make up a lune of angle equal to the given angle, and each vertex of this lune is a vertex of one of the solution triangles. In the case of one solution the given side lies along the polar of the vertices of the lune and must, therefore, be a special right spherical triangle in which the two opposite parts given must have like values. (See Figure 183.)

Classification: An ambiguous right spherical triangle will have:

1. ***Two*** solutions, if the given side is in the same quadrant as the given angle opposite, and if the value of the given side is farther from 90° than is the value of the given opposite angle;

2. ***One*** solution, if the value of the given side is exactly equal to the value of the given opposite angle;

3. ***No*** solution, if the value of the given side is nearer to 90° than is the value of the given opposite angle.

3. The A.S.S. Case: Geometrical Description and Classification of Solutions

Definition: *A great circular arc will be said to* ***lie within*** *a spherical angle, when the extremities of the arc lie on the sides of the angle and at distances less than* 180° *from the vertex of the angle.*

In Figure 184 the arc AD lies within the spherical angle C, and the arc AD' of the great circle of AD lies within the angle C'.

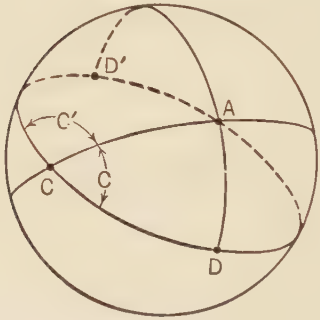


Figure 184

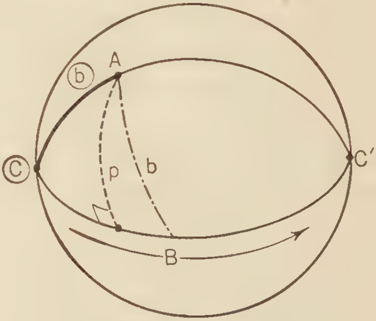


Figure 185

Convention concerning primed letters: A pair of identical letters, one unprimed and the other primed, will indicate either (1) *diametrically opposite points* on the sphere when the letters (capitals) represent *points*, or (2) *supplementary values* when the letters (capitals for spherical angles, small letters for great-circular arcs) represent *angular values*. In the case of the perpendiculars p and p', but not necessarily in any other cases, the primed value will always indicate the larger of the two supplementary values.

The following describes the procedure in investigating a.s.s. ambiguous triangles:

1. The lune of the given angle, C, is drawn and the given side, b, adjacent to C, is marked off from the vertex C of the lune and on one of its sides, to fix the vertex A. (See Figure 185.)

2. The variation in the length of the arc from A to a point B on the other side of the lune is then investigated as B moves continuously from C to C'. Since the perpendicular arc from A to the other side of the lune has an extreme value for all such arcs, it should be sketched in, as well as the arc b or b', whichever *lies within angle* C. The Principle of Continuous Variation of these arcs, stated below, will then indicate between what limits these arcs lie, depending on the positions of B.

3. The results of (2) will then show the limits on the possible values of the second given side, c, between which limits this second given side, together with the other two given parts, will form no triangle, one triangle, or two triangles. These limits on c will be found to depend on the magnitude of the first given side and the magnitude of the altitude which lies within the given angle C. This altitude is immediately determined from the given values of C and b, by Napier's Rules.

Principle of Continuous Variation in Arc Length from a Given Point to a Given Great Circle: *The variation in the length of a great-circular arc between a given point and a continuously moving point on a given great circle is continuous and lies between* p *and* p′, *the lengths of the perpendiculars from the given point to the given great circle.*

Let A and a in Figure 186 be the given point and great circle, respectively. The principle is obvious for A on a. Likewise, if A is a pole of a, the principle follows immediately, since, by Introduction, 6 e, all the arcs are quadrants and the variation is continuous because constant. If A is not a pole of a, let p be one of the two

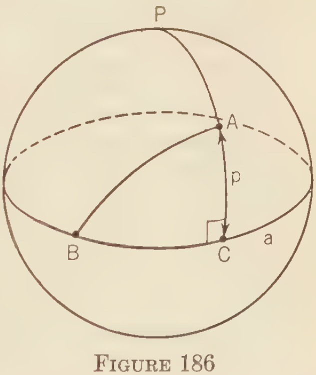


Figure 186

perpendiculars from A to a, and let its foot be C. Then, as the point B moves continuously along a

$$\cos AB = \cos p \cos BC, \text{ by Napier's Rules.}$$
$$AB = \cos^{-1}[(\cos p) \cos BC].$$

Hence, AB is a continuous function of a continuous argument.

The a.s.s. ambiguous triangles can best be investigated by treating each of four general cases individually:

1. C acute, b acute.
2. C acute, b obtuse.
3. C obtuse, b acute.
4. C obtuse, b obtuse.

The two special cases:

5. $C = 90°$
6. $C \neq 90°$, $b = 90°$,

are easily disposed of.

The investigations for all types are entirely similar to one another. The investigation for the general type (3) is given below. A concise statement of classification for all types follows.

3. C *obtuse,* b *acute:*

By Napier's Corollary 1 (see Figure 187), the perpendicular within C is p'. Let the foot of this perpendicular be D'. By Napier's Corollary 3 a, p' must lie inside one and outside the other of any possible *pair* of solutions arising from one given value of c.

As the vertex B moves continuously from C toward C' (and so that c lies within the angle C), the side c must continuously increase from the value of b (vertex B at C) to that of p' (vertex B at D') and then continuously decrease from the value of p' to that of b' (vertex B at C'), by Napier's Corollary 3 and the Principle of Continuous Variation.

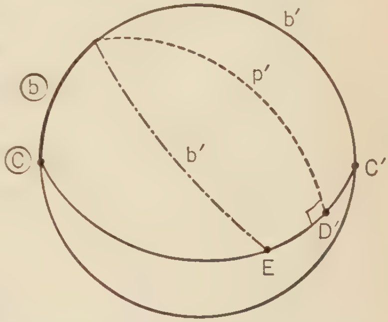


FIGURE 187

By the fact that b is given acute and by Napier's Corollary 3, $p' > b' > b$. Consequently, by Napier's Corollary 3 and by the fact that a continuous function must take on all values between any two that it takes on, the value of c must pass through that of b' as B moves from C to D'. Let the point between C and D' at which c equals b' be E.

Then, for B between E and C', c takes on every value between b' and p', once for B between E and D' and once for B between D' and C′. Hence, for c of any value between b and b' only one solution is possible (B between C and E); for c of any value between b' and p' there are two solutions possible (the vertices B_1 and B_2 straddling D'); for c equal to either b' or p' there will be but one solution possible (B at E or D', respectively) and the unique solution in the latter case will be a right triangle; for c any other value there will be no solution possible.

The results for this type can be summarized:

1. If the given value of c is between b' and p', there are two solutions. The perpendicular, p', lies inside one solution triangle and outside the other.

2. If the given value of c either (a) equals an end value of the interval described above for double solutions or (b) lies between b and b', there is one and only one solution.

3. If the given c has any other value, there is no solution.

By investigating the other types of a.s.s. Ambiguous Triangles the student can readily verify the following summary:

Classification of the Kinds of Solutions in the Ambiguous A.S.S. Case: Let the given angle and side adjacent be C and b, respectively. By Napier's Rules compute the perpendicular, p or p', from A and lying within angle C. Let $\{b\}$ be either b or b', whichever is in the quadrant of the evaluated perpendicular. Then

1. If the given value of c lies between the value of the perpendicular and $\{b\}$, there are ***two solutions.*** The altitude from A lies inside one and outside the other solution triangle.

2. If the given value of c either (a) equals an end value of the interval described above for possible double solutions or (b) lies in the interval bounded by b and b', there is ***one and only one solution.*** This single solution may be either a right or an isosceles triangle but need not be either.

3. If the given side c has any other value, there is ***no solution.***

4. The S.A.A. Case: Geometrical Description and Classification of Solutions

The geometric representation and algebraic classification of ambiguous solutions are much more difficult in the s.a.a. case than in the a.s.s. case. This may well be due to the absence of a plane analogue for the spherical s.a.a. case. The derivations of the classification in the s.a.a. case require several new theorems which are derived below.

Theorem 1: *If a particular s.a.a. case has a double solution, then the altitude necessary for the right-triangle-solution will lie* ***inside*** *or* ***outside both*** *solution triangles according as the two given angles are in the* ***same*** *or* ***different*** *quadrants.*

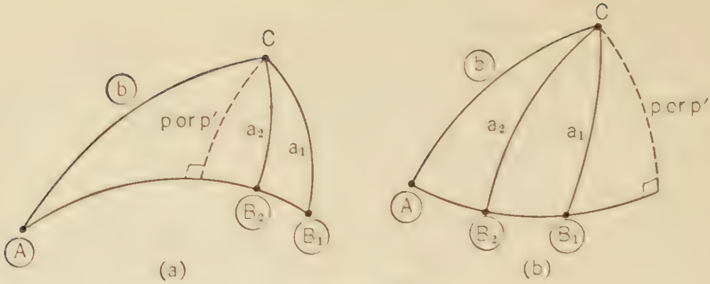


FIGURE 188

The statement and derivation of this theorem have been given in section 27. The derivation depends solely on Napier's Corollary 1, as Figure 188 indicates.

DEFINITION: *A great-circular arc of angular measure equal to 90° is* **a quadrant arc** *and is represented by* q *or* q′.

All great-circular arcs between a point and its polar are quadrant arcs by Introduction, 6 *e*.

CONSTRUCTION: *To construct the two quadrant arcs,* q *and* q′, *between a given point and a given great circle not the polar of the point.*

Let b in Figure 189 be the great circle through the given point, A, and the pole, P, of the given great circle, a. Let Q be the pole of B. Then Q lies on a, by Introduction, 6 *i*. Let the second intersection of the great circle of Q and A with a be Q'. Then $AQ = AQ' = 90°$.

CONSTRUCTION COROLLARY: *If between a given point and a given great circle the great circle of the perpendiculars and the great circle of the quadrant arcs are drawn, then*

1. *The four right triangles formed with a vertex at the given point are special right triangles, and the constructed great circles intersect perpendicularly at the given point.*

2. *The intersections of the two constructed great circles with the given great circle are 90° of arc apart.*

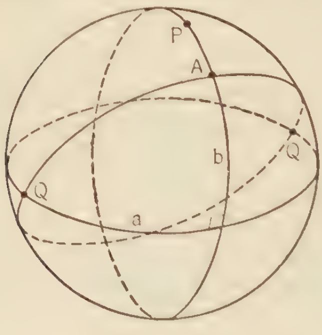


FIGURE 189

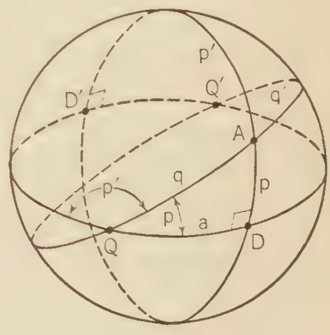


FIGURE 190

3. *The great circle of the quadrant arcs intersects the given great circle at two angles, each equal to the arc length of the constructed perpendicular arc which lies within that angle.*

In Figure 190 the given point and given great circle are A and a respectively, and the constructed perpendiculars and quadrant arcs are p and p' and q and q' respectively. The proofs of statements 1, 2, and 3 are immediately seen by the application of theorems 2 and 1 of section 17 (on special right triangles) to the construction, described above, of the quadrant arcs from a given point to a given great circle.

LEMMA 1: (**Polar analogue of Napier's Corollary 3 a**). *If two arcs (not on the same great circle) between a given point and a given great circle are supplementary and unequal, they must straddle * one of the quadrant arcs between the point and the great circle.*

By the principle of continuous variation of arcs between points and great circles (see the previous section), since continuous functions take on all values between any two taken on, and since 90° lies between two supplementary and unequal values, a quadrant arc must lie between the given unequal supplementary arcs b and b' connecting the point A with the great circle a. (See Figure 191.) As there are two quadrant arcs — on the same great circle — between A and a, one of these, q, must lie in that angle formed by b and b' which is less than 180°.

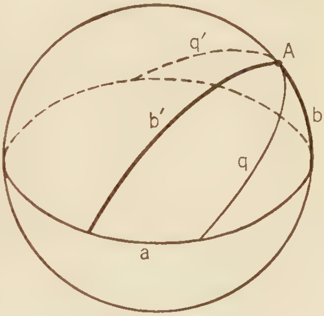


FIGURE 191

LEMMA 2: *In the s.a.a. Ambiguous Case, according as the given side,* b, *and the given adjacent angle,* A, *are in the* **same** *or* **different** *quadrants, the quadrant arc,* q, *from* C *and lying within angle* A, *must be* **farther from** *or* **nearer to** *angle* A *than is that perpendicular from* C *which lies within angle* A.

If angle A is acute (see Figure 192), the perpendicular will be the shorter one, p. If b is also acute, those arcs connecting C and side c which lie between b and p have values between the values of b and p — because the two perpendiculars, p and p', are the only extreme arcs between C and side c — and must, therefore, be acute. Hence, $q = 90°$ will lie farther from A than will the perpendicular p. However, if b is obtuse, $q = 90°$ must lie between the obtuse b and the acute perpendicular p, by the principle of continuous variation.

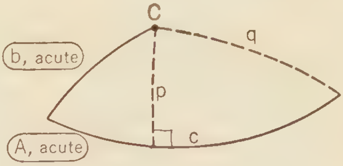


FIGURE 192

The reasoning for angle A obtuse is entirely similar.

* See the definition of "straddle" in section 12.

THEOREM 2: *If a particular s.a.a. case has a double solution, then the altitude necessary for the right-triangle-solution will lie* ***inside*** *or* ***outside*** *both solution triangles, according as the given side and given adjacent angle are in the* ***same*** *or* ***different*** *quadrants.*

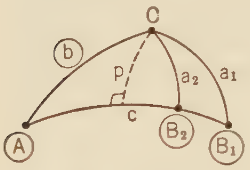


FIGURE 193

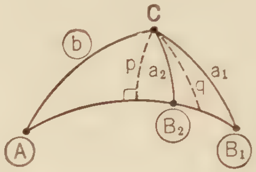


FIGURE 194 (*a*)
Side *b* same quadrant as angle *A*

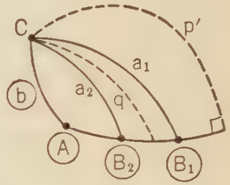


FIGURE 194 (*b*)
Side *b* different quadrant from angle *A*

From Figure 193 and by Napier's Rules the two values for side a are to be found from the sine.

$$\sin p = \sin a \sin B,$$
and
$$\sin a = \sin p \csc B.$$

These two values for a cannot be *equal*, because then the sides a, by Napier's Corollary 3 a, would straddle the perpendicular, p or p', from C. This is impossible, by theorem 1. Consequently, the two values for side a must be *supplementary* and unequal for two solutions.

Hence, by lemma 1, the two sides a_1 and a_2 must straddle q, but not p.

Then, by lemma 2, according as the given side b and the given adjacent angle A are in the same quadrants or in different quadrants, the two possible positions of the unknown side a, opposite the given angle A, are both farther from or both nearer to the vertex A than is the perpendicular, p. (See Figure 194 (a) and (b).)

THEOREM 3: *If a particular s.a.a. case is to have a double solution, the given opposite parts must lie in the same quadrant.*

We now have two criteria for the position of the perpendicular from the vertex of the unknown angle: that of theorem 1, depending on whether the two given angles are in the same quadrant or in different quadrants, and that of theorem 2, depending on whether the given side and given angle adjacent to this side are in the same quadrant or in different quadrants.

Suppose, first, that the given angle adjacent to the given side is in the same quadrant as the other angle: Then, by theorem 1, the perpendicular, p, from the vertex of the unknown angle will lie ***inside*** both triangle solutions. Now then, if the given side were not in the same quadrant as the given angle opposite it, this given side would not be in the same quadrant as its adjacent angle and, therefore, by theorem 2, p would lie ***outside*** both triangle solutions, which contradicts the above supposition.

The proof for the given angle adjacent to the given side not in the same quadrant as the other given angle is entirely similar.

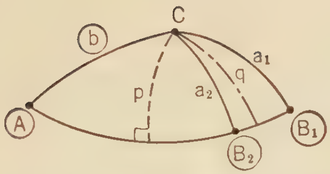


FIGURE 195

THEOREM 1: *Given angles in same quadrant.*

THEOREM 2: *Given side and given adjacent angle in same quadrant.*

THEOREM 3: *Given side and given opposite angle in same quadrant.*

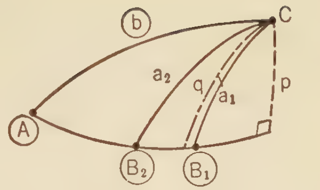


FIGURE 196

THEOREM 1: *Given angles in different quadrants.*

THEOREM 2: *Given side and given adjacent angle in different quadrants.*

THEOREM 3: *Given side and given opposite angle in same quadrant.*

Figures 195 and 196 picture the results of the above three theorems for the cases in which the s.a.a. triangles have double solutions.

PRINCIPLE OF CONTINUOUS VARIATION IN THE ANGLE BETWEEN A GIVEN GREAT CIRCLE AND GREAT-CIRCULAR ARCS THROUGH A GIVEN POINT. *The angle at which a variable arc, between a given point and a variable point on a given great circle, intersects the given great circle varies continuously between the extreme angles* p *and* p′, *the angular lengths of the perpendicular arcs between the given point and given great circle. The extreme values are taken on at points 90° of arc from the feet of the perpendiculars from the given point to the great circle. By a property of continuous functions, all values of the angle between those for any two positions of the vertex must be taken on as the vertex moves continuously from one to the other of these two positions.*

In Figure 197 the given point and given great circle are A and a, respectively. D and D' are the intersections of p and p', respectively, with a, and Q and Q' are the intersections of q and q', respectively (the quadrant arcs from A to a), with a.

Point B is considered to move to the left and angle B is taken as angle ABD, *where* BD *is taken opposite to the motion of* B. In this way angle B is kept on the same side of the arc AB.

By Napier's Rules:

$$\sin BD = \cot B \tan p$$
$$B = \cot^{-1}[(\cot p)\sin BD]$$

where $\cot p$ is constant.

For the point B on DQD' $\sin BD$ varies continuously between 0 (B at D), 1 (B at Q), and 0 (B at D'), and, therefore, angle

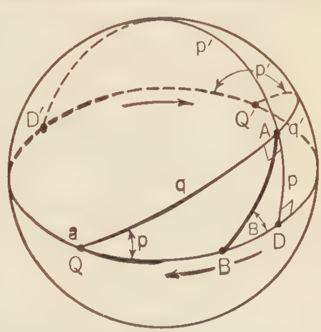


FIGURE 197

B varies between 90°, p, and 90°, respectively, since the arc cotangent function is continuous between 0 and cot p.

By the same reasoning for right triangles to the right of the arc p, angle B' varies between 90° (point B at D'), p (point B at Q'), 90° (point B at D). Consequently in this second half of the great circle a, the angle B varies from 90° (point B at D') to p' (point B at Q') to 90° (point B at D again).

The geometrical description and classification of s.a.a. triangles can now be investigated. The procedure for all general cases, neither the given side nor its given adjacent angle being equal to 90°, is as follows. (See Figure 198.)

1. Draw the lunė determined by the given angle, A, adjacent to the given side, b.

2. From the vertex A of the lune and along one of its sides lay off the given side b to fix the vertex C of the required triangle or triangles.

3. Let the vertex B move from A to A', continuously along that side of the lune which is opposite C, and note the variation in the angle B of the triangle ABC. In this process the principle of continuous variation of angle between a given great circle and great-circular arcs through a given point is most useful. Furthermore, the three auxiliary arcs *from* C *and lying within angle* A:

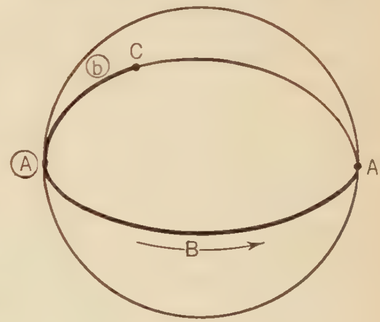


FIGURE 198

a. the unique perpendicular, p or p', meeting the side of c in D or D', respectively,

b. the unique quadrant arc, q, meeting c in Q, and

c. the unique arc b or b' meeting c in $\overline{B}$ or $\overline{B}'$,

should be sketched and their several properties, hitherto discussed, recalled. Accordingly, the relative positions of A, D or D', $\overline{B}$ or $\overline{B}'$, Q, and A' on the side of c will be known. (See Figure 199.)

4. On the basis of the investigations in (3) and by means of theorems 1, 2, and 3 of this section, together with he propertiets of the three auxiliary arcs sketched from C to c, the types of possible solutions (and their geo-

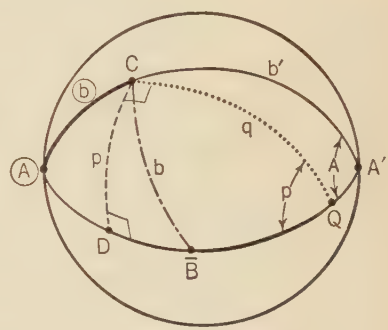


FIGURE 199

metrical representations) can be classified *according to the value of the given second angle*, B.

The above procedure has been applied to the four combinations which together include all the general cases with results indicated below. To describe the procedure further a more complete explanation is given in the first and third combinations. Finally, the results of all cases are summarized in one concise statement.

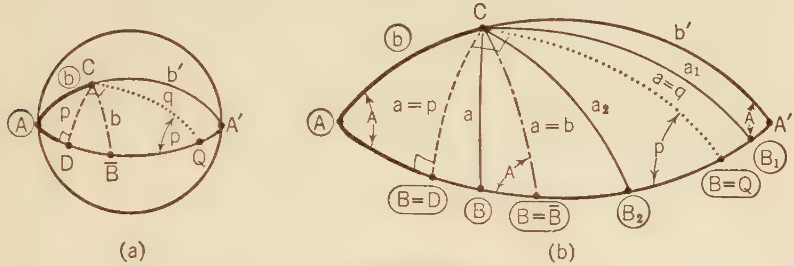
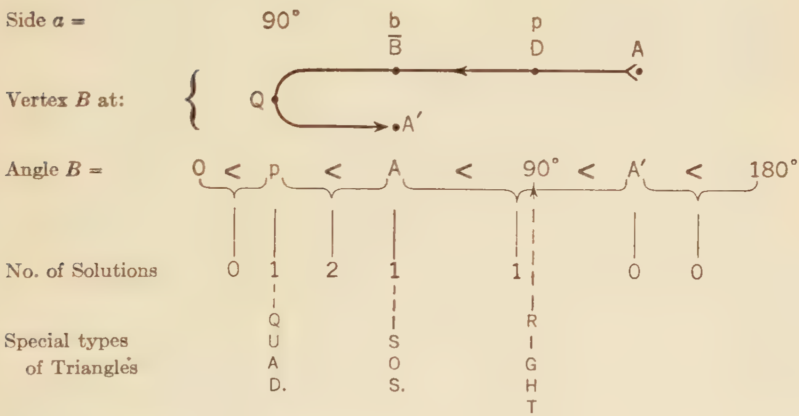


FIGURE 200



First Combination: A *acute*, b *acute; Figure* 200.

As *vertex* B moves from A to D, *angle* B decreases from A' to 90°.
As *vertex* B moves from D to $\bar{B}$, *angle* B decreases from 90° to A.
As *vertex* B moves from $\bar{B}$ to Q, *angle* B decreases from A to p.*
As *vertex* B moves from Q to A', *angle* B increases from p to A.

This variation in angle B according to the position of vertex B is indi-

* Note that, by Napier's Rules,

$$\sin p = \sin b \sin A.$$

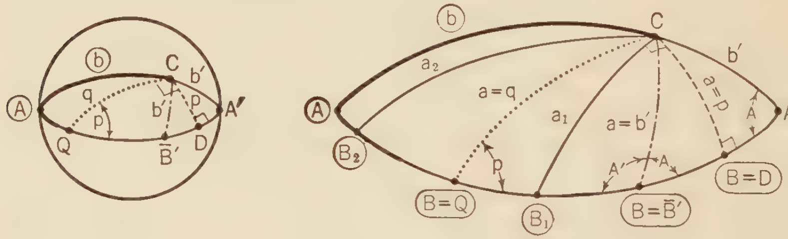
Hence, $\sin p$ is never greater than $\sin A$. Therefore,

$$p < A,\ A \textit{ acute};\ p' > A,\ A \textit{ obtuse}.$$

cated by the curved arrow in the accompanying diagram. The range of values of angle B *which are taken on twice*, that is, the range over which the arrow showing the motion of vertex B *curves back*, is the range of values of angle B which will give *double solutions*.

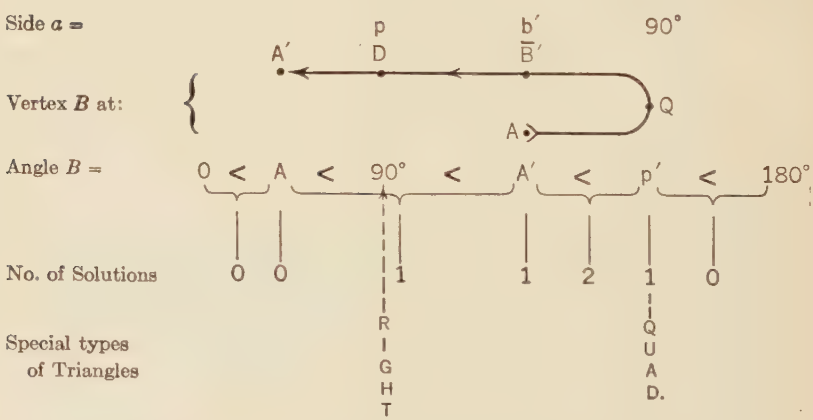
The results of the diagram for this first combination can be summarized:

1. There will be ***two solutions*** if the value of the given second angle, B, lies between the value of angle A and the value of the (acute) perpendicular, p, from C lying within A.
2. There will be ***one solution*** if the value of the given angle B lies (a) between A and A', or (b) at either extremity of the above range for double solutions.
3. There will be ***no solution*** for any other value of B.



(a) (b)

FIGURE 201



Second Combination: A *acute*, b *obtuse; Figure* 201.

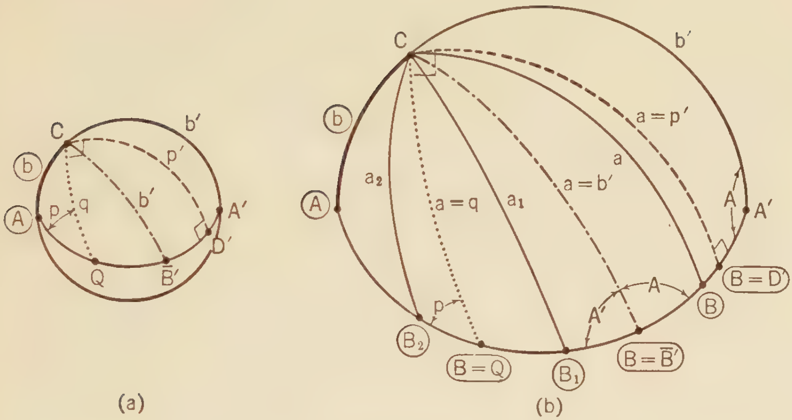
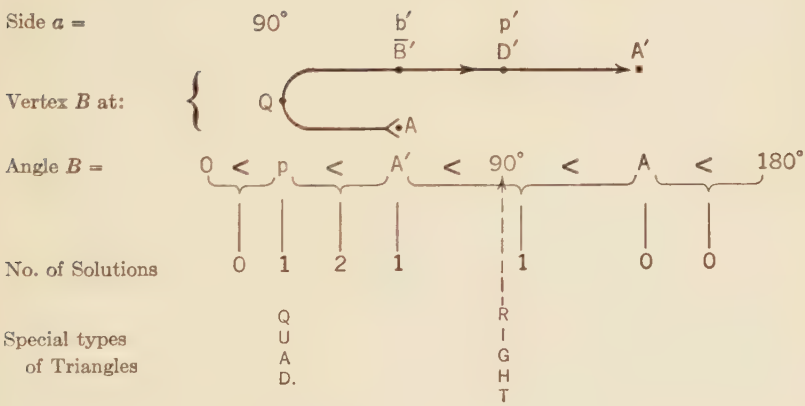


FIGURE 202



Third Combination: A *obtuse*, b *acute; Figure* 202.

As *vertex* B moves from A to Q, *angle* B decreases from A' to p.
As *vertex* B moves from Q to $\overline{B}'$, *angle* B increases from p to A'.
As *vertex* B moves from $\overline{B}'$ to D', *angle* B increases from A' to 90°.
As *vertex* B moves from D' to A', *angle* B increases from 90° to A.

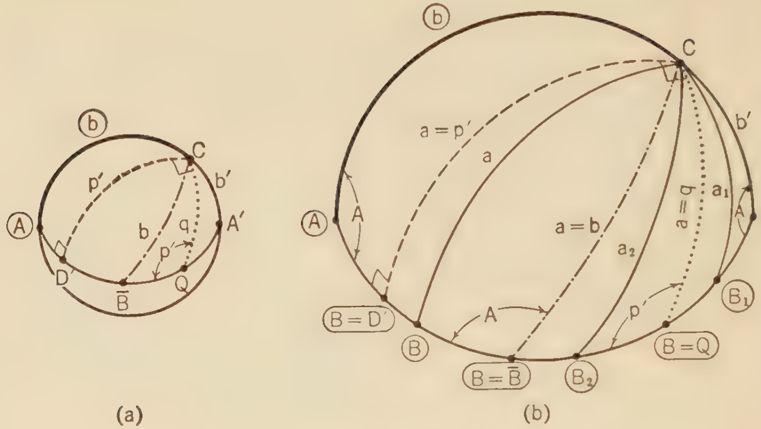
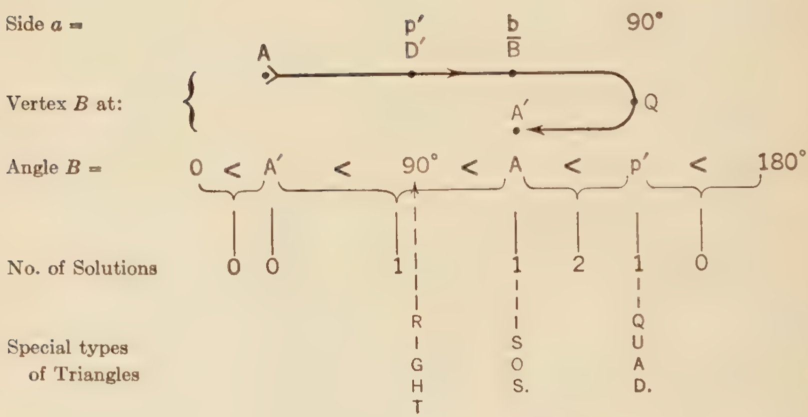


FIGURE 203



Fourth Combination: A *obtuse,* b *obtuse; Figure* 203.

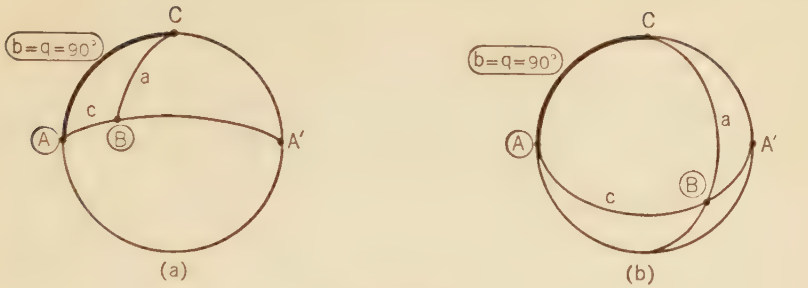


FIGURE 204

Special Combinations:

a. $A = 90°$:

The triangle is then an ambiguous *right* triangle, a type discussed in section 16.

b. $A \neq 90°, b = 90°$: (See Figure 204.)

Since a quadrant arc from C does not lie within the lune of angle A, double solutions are impossible, by lemma 1. Single solutions are obviously possible for angle B between the values A and A', exclusive of the end values.

Observation of all of the combinations of the s.a.a. ambiguous case will immediately verify the following.

Classification of types of solutions in the ambiguous s.a.a. case. Let the given side and adjacent angle be b and A, respectively. By Napier's Rules compute the perpendicular, p or p' from C lying within angle A. Let $\{A\}$ be either A or A', whichever is in the quadrant of b. Similarly, let $\{p\}$ be either p or p', whichever is in the quadrant of b. Then

1. If the given value of B lies between $\{A\}$ and $\{p\}$, there are ***two solutions.*** The altitude from C lies inside both solution triangles for B in the quadrant of A and outside both solution triangles for A and B in different quadrants. Furthermore, the sides opposite A in the two solution triangles straddle the quadrant arc from C.

2. If the given value of B either *(a)* equals an end value of the above described interval for possible double solutions, or *(b)* lies in the interval A, A', there is ***one and only one solution.*** The altitude from C lies inside or outside the triangle according as B is or is not in the quadrant of A.

3. If the given angle B has any other value, there is ***no solution.***

Alternate Methods of Spherical Triangle Solution

5. Critique

The following sections deal with the derivations of formulas by which general spherical triangles can be solved without dividing them into two right triangles to be solved by Napier's Rules. For certain given triangles some of these methods will be simpler than the right triangle method. The labor involved in first deriving and then remembering these formulas, however, outweighs any time which may be saved in the numerical calculation of certain problems by these special methods instead of by the one fundamental right triangle method. Furthermore, the right triangle method never leads to uncertainties, whereas, for instance, the law of sines method always leads to two solutions, and the spurious solution often cannot be discarded by the law of magnitude relation. In such a case an additional method must be invoked, making the actual computation more extensive than the computation by the right triangle method.

These special methods are here briefly derived for those who may prefer them as alternate methods for certain triangles. They may be useful also for the light which they can throw on some of the specialized systems of navigation by which the professional navigator, having previously been grounded in the theory of spherical trigonometry, solves certain frequently occurring types of triangles largely by tables.* In an attempt to make these derivations less blind, full use is made of any existing similarities with corresponding plane trigonometry laws. It is hoped that, since the plane laws are familiar and relatively simple, the spherical derivations will therefore be not merely proofs of arbitrarily stated truths but explorations of unknown but desired relationships.

* See, for example, *Ageton's Method* (devised by Commander A. A. Ageton), *Dreisenstok's Method* (devised by Commander J. Y. Dreisenstok), and *Weems's Method* (devised by Lieutenant Commander P. V. H. Weems).

6. The Law of Sines

In any spherical triangle the ratio of the sine of any side to the sine of the angle opposite is the same as the corresponding ratios for the other sides and angles of the triangle, or:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

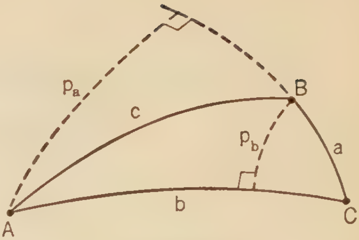


FIGURE 205

This law is markedly similar to the corresponding law in plane trigonometry in both statement and proof (compare Introduction, 22 *a*). It is proved by dropping altitudes onto two sides of the triangle and equating two expressions for each altitude by Napier's Rules in the right triangles thus formed (see Figure 205).

$$\sin p_b = \sin c \sin A = \sin a \sin C.$$

Therefore,

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}.$$

$$\sin p_a = \sin c \sin (\pi - B) = \sin b \sin C,$$

or

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Therefore,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Because unknowns are found from the sine or cosecant functions, this law leads to uncertainties which the law of magnitude of parts of a triangle may not be able to dispel. See example 1 on page 199 for an application of the law of sines to a particular triangle.

7. The Law of Cosines for Sides

In any spherical triangle the cosine of any side equals the product of the cosines of the other two sides increased by the product of the sines of these other two sides and the cosine of the angle included by these other two sides, or

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

and similarly for the sides b *and* c.

Knowing of the existence in plane trigonometry of a law expressing an

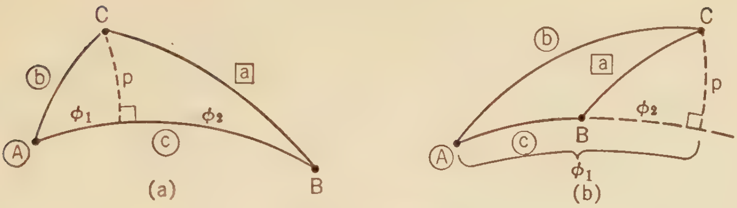


FIGURE 206

unknown side in terms of the other two sides and their included angle, one is led to explore by analogy a corresponding law in spherical trigonometry. Reference to Introduction, 22 b will justify the reasonableness of the following attack (see Figure 206):

1. Drop an altitude onto a known side and express by Napier's Rules — instead of by the Pythagorean theorem as in plane trigonometry — the unknown side in terms of this altitude and a ϕ.

$$\cos a = \cos p \cos \phi_2$$

2. Express ϕ_2 in terms of a known side and ϕ_1 — as in plane trigonometry.

$$\phi_2 = \pm (c - \phi_1)$$
$$\cos a = \cos p \cos c \cos \phi_1 + \cos p \sin c \sin \phi_1$$

3. The desire to eliminate the two unknown auxiliaries, ϕ_1 and p, and to introduce at the same time the other known side suggests writing the Napier's Rule formulas involving all three of these parts.

$$\cos b = \cos \phi_1 \cos p$$
$$\cos a = \cos b \cos c + \cos p \sin c \sin \phi_1$$

4. By analogy with the plane formula we now wish to introduce the known angle and eliminate the remaining factors involving an auxiliary. This suggests a Napier's Rule formula involving A and one or both of the auxiliaries involved by their respective functions in the derivation to date. Simple experimentation will result in the obvious choice.

$$\sin \phi_1 = \cot A \tan p$$
$$\cos a = \cos b \cos c + \cos p \sin c \cot A \tan p$$
$$\cos a = \cos b \cos c + \sin c \cot A \sin p$$

5. Eliminating the remaining auxiliary in favor of known parts suggests a unique Napier's Rule formula.

$$\sin p = \sin b \sin A$$
$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

q.e.f.

If, instead of a derivation, that is, a motivated exploration of an unknown relation whose need is felt in a general way, a proof of the cosine law, previously and arbitrarily stated, is desired, the following geometric analysis will be found to be shorter than the synthesis given above.

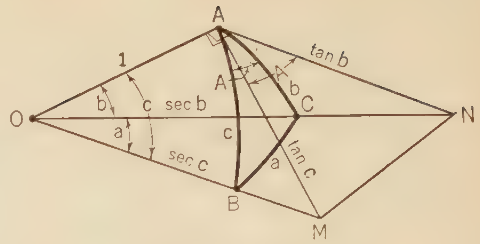


FIGURE 207

The vertices of the spherical triangle ABC (see Figure 207) are connected to the center of the sphere and tangents at A to the sides b and c are drawn, meeting OB and OC extended in M and N, respectively. The plane angle MAN equals the spherical angle A by Introduction, 6 b. The evaluations of the segments follows from Introduction, 9 b and 11 b. Then, use the law of cosines in plane trigonometry (Introduction, 22 b) on the plane triangles OMN and MAN to evaluate the square of MN.

$$MN^2 = \sec^2 b + \sec^2 c - 2 \sec b \sec c \cos a$$
$$MN^2 = \tan^2 b + \tan^2 c - 2 \tan b \tan c \cos A$$

Subtracting, $0 = 1 + 1 - 2\dfrac{\cos a}{\cos b \cos c} + 2\dfrac{\sin b \sin c \cos A}{\cos b \cos c}$,

by Introduction, 14. Then

$$0 = 2 - 2\frac{\cos a - \sin b \sin c \cos A}{\cos b \cos c}$$
$$0 = 2 \cos b \cos c - 2 (\cos a - \sin b \sin c \cos A)$$
$$\cos a = \cos b \cos c + \sin b \sin c \cos A; \textit{ q.e.d.}$$

See example 1 on page 199 for applications of the law of cosines to particular triangles.

8. The Law of Cosines for Angles

$$\cos \text{A} = -\cos \text{B} \cos \text{C} + \sin \text{B} \sin \text{C} \cos \text{a},$$
and similarly for angles B *and* C.

This law naturally follows from "polarizing" the law of cosines for sides. First write this law for the polar triangle of the given triangle ABC using, as usual, primes to indicate supplements or corresponding polar parts.

$$\cos a' = \cos b' \cos c' + \sin b' \sin c \cot A'$$
Then, polarize.

$$\cos(\pi - A) = \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a)$$
$$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C (-\cos a)$$
$$\cos A = -\cos B \cos C + \sin B \sin C \cos a; \textit{ q.e.d.}$$

The two cosine laws, because they involve sums and differences instead of products and quotients, are not easily adaptable to logarithmic computation.

9. The Half-Angle Formulas

1. *For angles:*

$$\tan\tfrac{1}{2}A = \frac{r}{\sin(s-a)},\ \tan\tfrac{1}{2}B = \frac{r}{\sin(s-b)},\ \tan\tfrac{1}{2}C = \frac{r}{\sin(s-c)}$$

where

$$s = \tfrac{1}{2}(a+b+c)\ \textit{and}\ r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

2. *For sides:*

$$\cot\tfrac{1}{2}a = \frac{R}{\cos(S-A)},\ \cot\tfrac{1}{2}b = \frac{R}{\cos(S-B)},\ \cot\tfrac{1}{2}c = \frac{R}{\cos(S-C)}$$

where

$$S = \tfrac{1}{2}(A+B+C)\ \textit{and}\ R = \sqrt{\frac{\cos(S-A)\cos(S-B)\cos(S-C)}{-\cos S}}.$$

1. These formulas, which are analogous to the similarly named formulas in plane trigonometry for the solution of the plane s.s.s. case, are useful in the logarithmic solution of this case in spherical trigonometry. Their derivations are also analogous to those of the plane trigonometry formulas and are roundabout transformations of the cosine laws.

Since a formula is desired for an angle in terms of the three sides, it should be helpful — just as in the plane case — to solve a law-of-cosines-formula-for-sides for an angle in terms of functions of the three sides.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

Now, exactly as in the plane case (compare Introduction, 22 c), by forming $1 - \cos A$ and $1 + \cos A$, on the one hand, the half angle formulas will apply, and on the other hand, algebraic and trigonometric transformations lead to products which are more useful than sums and differences for logarithmic computation.

$$2\sin^2\tfrac{1}{2}A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

$$2\cos^2\tfrac{1}{2}A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

$$2 \sin^2 \tfrac{1}{2} A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c}$$

and $$2 \cos^2 \tfrac{1}{2} A = \frac{\sin b \sin c - \cos b \cos c + \cos a}{\sin b \sin c}.$$

Hence, $$2 \sin^2 \tfrac{1}{2} A = \frac{-2 \sin \frac{a+b-c}{2} \sin \frac{-a+b-c}{2}}{\sin b \sin c};$$

$$2 \cos^2 \tfrac{1}{2} A = \frac{-2 \sin \frac{a+b+c}{2} \sin \frac{a-b-c}{2}}{\sin b \sin c}.$$

As in plane trigonometry, let $s = \frac{1}{2}(a+b+c)$.

Then, $$\frac{a+b-c}{2} = \frac{a+b+c}{2} - \frac{2c}{2} = s - c, \text{ etc.};$$

and therefore,

$$2 \sin^2 \tfrac{1}{2} A = \frac{-2 \sin (s-c) \sin (-s+b)}{\sin b \sin c};$$

$$2 \cos^2 \tfrac{1}{2} A = \frac{-2 \sin s \sin (a-s)}{\sin b \sin c}.$$

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-c) \sin (s-b)}{\sin b \sin c}};$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}.$$

The plus sign alone is possible, because for A, an angle of a spherical triangle, $\frac{1}{2} A$ must be first quadrant.

Consequently, $$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}}.$$

Writing the right-hand side of the above as

$$\frac{1}{\sin (s-a)} \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}$$

and letting $$r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}},$$

the above formula becomes

$$\tan \tfrac{1}{2} A = \frac{r}{\sin (s - a)}.$$

Since A is any angle of a spherical triangle,

$$\tan \tfrac{1}{2} B = \frac{r}{\sin (s - b)} \text{ and } \tan \tfrac{1}{2} C = \frac{r}{\sin (s - c)}.$$

If but one angle is required, the cosine formula requires less computation. If more than one is to be found, it is best to use the tangent formulas.

2. Analogous formulas for functions of half the sides in terms of the three angles can be derived for the a.a.a. case in spherical trigonometry by polarizing the formulas given above for half the angles in terms of the three sides. (The analogous formulas, of course, do not exist in plane trigonometry, because the a.a.a. case in plane trigonometry is indeterminate.) Or, the procedure in the derivation given above can be repeated beginning with the law of cosines for angles:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

$$2 \sin^2 \tfrac{1}{2} a = \frac{-\cos A - \cos B \cos C + \sin B \sin C}{\sin B \sin C} = \frac{-\cos A - \cos (B + C)}{\sin B \sin C}$$

and

$$2 \cos^2 \tfrac{1}{2} a = \frac{\cos A + \cos B \cos C + \sin B \sin C}{\sin B \sin C} = \frac{\cos A + \cos (B - C)}{\sin B \sin C}.$$

Therefore,

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}}, \quad \cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}},$$

$$\text{and } \cot \tfrac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{-\cos S \cos (S - A)}} =$$

$$\frac{1}{\cos (S - A)} \sqrt{\frac{\cos (S - A) \cos (S - B) \cos (S - C)}{-\cos S}}.$$

Finally, letting

$$R = \sqrt{\frac{\cos (S - A) \cos (S - B) \cos (S - C)}{-\cos S}},$$

and recognizing that a is any side of a spherical triangle, we have

$$\cot \tfrac{1}{2} a = \frac{R}{\cos (S - A)}; \quad \cot \tfrac{1}{2} b = \frac{R}{\cos (S - B)}; \quad \cot \tfrac{1}{2} c = \frac{R}{\cos (S - C)}.$$

Since the sum of the angles of a spherical triangle must lie between 180°

and 540°, S must lie between 90° and 270°, which means that the cosine of S will be negative to insure the positiveness of the radicand.

See example 2 on page 200 for applications of the half-angle formulas to particular triangles.

10. Napier's Analogies

1. *For angles:*

$$\frac{\tan \frac{1}{2}(A - B)}{\cot \frac{1}{2} C} = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \text{ and } \frac{\tan \frac{1}{2}(A + B)}{\cot \frac{1}{2} C} = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}$$

and similar pairs of formulas involving a *and* c, *and* b *and* c.

2. *For sides:*

$$\frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2} c} = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \text{ and } \frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2} c} = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)}$$

and similar pairs of formulas involving A *and* C, *and* B *and* C.

The formulas for angles are obviously "analogous" to the law of tangents in plane trigonometry, discussed in Introduction, 22 *d*. The formulas for sides are mere polarizations of the formulas for angles.

In logarithmic solutions of the s.a.s. and the a.s.a. cases, a suitable *pair* of formulas from these two sets — that is, two formulas involving the same five parts of a spherical triangle — is *convenient*. The use of two formulas is illustrated in example 4 on page 202.

In the solution of the a.s.s. and s.a.a. ambiguous cases, *individual* formulas from these two sets are *essential*.* Any one of these formulas can be solved for one part of a spherical triangle in terms of two pairs of opposite parts. Given one pair of opposite parts and a third part, the two pairs of opposite known parts can be supplied by an application of the law of sines. The use of the law of sines and of an individual Napier's analogies formula are needed for the solution of problem 6 on page 203.

In plane trigonometry oblique triangles can always be solved by the law of sines and the law of cosines. In spherical trigonometry, however, because the values of two angles of a spherical triangle do not determine the third angle, oblique triangles cannot always be solved by the law of sines and cosines. Hence, while in plane trigonometry the law of tangents is merely a logarithmic convenience, in spherical trigonometry Napier's analogies are a necessity.

1. Beginning with the law of sines (by analogy with the law of tangents in plane trigonometry):

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$$

* That is, when the division-into-right-triangles-method, recommended in the text proper for the solution of all oblique spherical triangles, is not used.

and, as in plane trigonometry, taking this proportion in subtraction and addition:

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\sin a - \sin b}{\sin a + \sin b}$$

$$\frac{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)} = \frac{2 \cos \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)}{2 \sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b)}$$

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}(a + b)}.$$

This is analogous in form to the law of tangents in plane trigonometry. This, however, is useless in spherical trigonometry for the solution of the s.a.s. case, because in spherical triangles — in contrast to plane triangles where the sum of the angles is constant — knowing angle C does not determine $(A + B)$. From this we conclude that the one formula above cannot be used to find $(A - B)$, but that *two formulas are needed, one for* $\frac{1}{2}$ (A − B) *and one for* $\frac{1}{2}$ (A + B), *each involving functions of* $\frac{1}{2}$ (a + b) *and* $\frac{1}{2}$ (a − b). These essentials, plus the fact that sines and cosines are more susceptible to formula manipulation than tangents and cotangents, might naturally suggest the following transformation on the above:

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{\dfrac{\sin \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a - b)}}{\dfrac{\sin \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a + b)}},$$

or

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{\dfrac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}}{\dfrac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}}.$$

From this, keeping in mind what is wanted: that is, a formula for $\frac{1}{2}(A + B)$ and another for $\frac{1}{2}(A - B)$, each involving $\frac{1}{2}(a + b)$ and $\frac{1}{2}(a - b)$; we can write

$$\tan \frac{1}{2}(A - B) = k \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}; \quad \tan \frac{1}{2}(A + B) = k \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}.^*$$

The problem now is to evaluate k. This quantity k is to satisfy two con-

* If $\frac{x}{y} = \frac{m}{n}$, then we can define k by $x = km$. Then y, which is equal to $x\frac{n}{m}$, is given by $y = km\frac{n}{m}$ or $y = kn$.

Hence, $\frac{x}{y} = \frac{m}{n}$ implies $x = km$, $y = kn$.

ditions. Multiplying together the two equations above gives an expression involving both conditions on k and should therefore lead to the evaluation of k.

$$\tan \tfrac{1}{2}(A - B) \tan \tfrac{1}{2}(A + B) = k^2 \frac{\sin \frac{1}{2}(a-b) \cos \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a+b)}$$

Replacing tangents by sines and cosines, since more formulas are available for these latter two functions, we get this equation:

$$\frac{\sin \frac{1}{2}(A-B) \sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B) \cos \frac{1}{2}(A+B)} = k^2 \frac{\sin \frac{1}{2}(a-b) \cos \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a+b)}$$

Perform the suggested transformations to obtain functions of each whole angle separately.

$$\frac{-\frac{1}{2}(\cos A - \cos B)}{\frac{1}{2}(\cos A + \cos B)} = k^2 \frac{\frac{1}{2}\sin(a-b)}{\frac{1}{2}\sin(a+b)}$$

Angles A and B can now be evaluated in terms of sides alone by means of the law of cosines for sides, thus evaluating k^2 in terms of sides alone.

$$k^2 = -\frac{\sin(a+b)}{\sin(a-b)} \cdot \frac{\dfrac{\cos a - \cos b \cos c}{\sin b \sin c} - \dfrac{\cos b - \cos a \cos c}{\sin a \sin c}}{\dfrac{\cos a - \cos b \cos c}{\sin b \sin c} + \dfrac{\cos b - \cos a \cos c}{\sin a \sin c}}$$

The remaining steps, though complicated looking, are immediately suggested by the natural desire for simplicity.

$$k^2 = -\frac{\sin(a+b)}{\sin(a-b)} \frac{\sin a \cos a - \sin a \cos b \cos c - \sin b \cos b + \sin b \cos a \cos c}{\sin a \cos a - \sin a \cos b \cos c + \sin b \cos b - \sin b \cos a \cos c}$$

$$= -\frac{\sin(a+b)}{\sin(a-b)} \frac{\frac{1}{2}\sin 2a - \cos c(\sin a \cos b - \cos a \sin b) - \frac{1}{2}\sin 2b}{\frac{1}{2}\sin 2a - \cos c(\sin a \cos b + \cos a \sin b) + \frac{1}{2}\sin 2b}$$

$$= -\frac{\sin(a+b)}{\sin(a-b)} \frac{\frac{1}{2}(\sin 2a - \sin 2b) - \cos c \sin(a-b)}{\frac{1}{2}(\sin 2a + \sin 2b) - \cos c \sin(a+b)}$$

$$= -\frac{\sin(a+b)}{\sin(a-b)} \frac{\frac{1}{2}\, 2\cos(a+b)\sin(a-b) - \cos c \sin(a-b)}{\frac{1}{2}\, 2\sin(a+b)\cos(a-b) - \cos c \sin(a+b)}$$

$$= -\frac{\cos(a+b) - \cos c}{\cos(a-b) - \cos c} = -\frac{-2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(a+b-c)}{-2\sin\frac{1}{2}(a+c-b)\sin\frac{1}{2}(a-b-c)}$$

$$= -\frac{\sin s \sin(s-c)}{\sin(s-b)\sin(a-s)} = \frac{\sin s \sin(s-c)}{\sin(s-a)\sin(s-b)}$$

$$= \frac{\sin s}{\sin(s-a)\sin(s-b)\sin(s-c)} \sin^2(s-c) = \frac{\sin^2(s-c)}{r^2}$$

Therefore, $k = \dfrac{\sin(s-c)}{r} = \cot \frac{1}{2} C$, by the previous section.*

* Of the two algebraic possibilities, $k = \pm \cot \dfrac{C}{2}$, only the plus sign is possible.

Therefore, $\tan \frac{1}{2}(A - B) = \cot \frac{1}{2} C \dfrac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}$,

and $\qquad \tan \frac{1}{2}(A + B) = \cot \frac{1}{2} C \dfrac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}$,

from which, by dividing by $\cot \frac{1}{2} C$, the desired formulas are obtained. *q.e.f.*

Since the parts of the spherical triangle represented in the pair of formulas above are any set of parts exhibiting the given relationships of "opposite" and "included," the formulas have been proved for all other similar sets of parts of the spherical triangle.

If two sides and the included angle of any spherical triangle are given, these formulas are convenient for the logarithmic evaluation of half the difference and half the sum of the other two angles. Adding and subtracting half the sum and half the difference will give these other two angles individually.

3. The desire to investigate polarizing the pairs of formulas above should be a natural one. Beginning with the above law applied to the polar triangle (so that the formula to be derived will apply to the given triangle):

$$\frac{\tan \dfrac{A' - B'}{2}}{\cot \dfrac{C'}{2}} = \frac{\sin \dfrac{a' - b'}{2}}{\sin \dfrac{a' + b'}{2}} \qquad \frac{\tan \dfrac{A' + B'}{2}}{\cot \dfrac{C'}{2}} = \frac{\cos \dfrac{a' - b'}{2}}{\cos \dfrac{a' + b'}{2}}$$

$$\frac{\tan \dfrac{\pi - a - (\pi - b)}{2}}{\cot \dfrac{\pi - c}{2}} = \frac{\sin \dfrac{\pi - A - (\pi - B)}{2}}{\sin \dfrac{\pi - A + (\pi - B)}{2}}; \qquad \frac{\tan \dfrac{\pi - a + \pi - b}{2}}{\cot \dfrac{\pi - c}{2}} = \frac{\cos \dfrac{\pi - A - (\pi - B)}{2}}{\cos \dfrac{\pi - A + \pi - B}{2}}$$

$$\frac{\tan - \frac{1}{2}(a - b)}{\tan \frac{1}{2} c} = \frac{\sin - \frac{1}{2}(A - B)}{\sin [\pi - \frac{1}{2}(A + B)]} \qquad \frac{\tan [\pi - \frac{1}{2}(a + b)]}{\tan \frac{1}{2} c} = \frac{\cos - \frac{1}{2}(A - B)}{\cos [\pi - \frac{1}{2}(A + B)]}$$

$$\frac{- \tan \frac{1}{2}(a - b)}{\tan \frac{1}{2} c} = \frac{- \sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \qquad \frac{- \tan \frac{1}{2}(a + b)}{\tan \frac{1}{2} c} = \frac{\cos \frac{1}{2}(A - B)}{- \cos \frac{1}{2}(A + B)}$$

$$\frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2} c} = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)}. \qquad \frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2} c} = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)}.$$

11. Some Haversine Laws

DEFINITION: *The* ***versed sine,**** *that is, "reversed sine," of an angle is one minus the cosine of the angle, or*

$$\text{vers } \theta = 1 - \cos \theta.$$

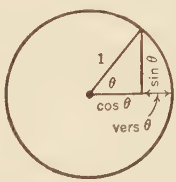


FIGURE 208

* The versed sine of a central angle is that part of the radius of the unit circle between the foot of the sine of the angle and the arc. (See Figure 208.)

Definition: *The **haversine** of an angle is half the versed sine of the angle, or*

$$hav\ \theta = \frac{1 - \cos\theta}{2}.$$

Theorem: $hav\ \theta = \sin^2 \frac{1}{2}\theta$.

The half-angle laws and the cosine laws can now be transformed to give formulas involving this haversine function. These haversine laws are sometimes used for solution of spherical triangles when the somewhat bulky haversine tables are available.* The haversine laws contribute nothing essentially new to the theory of spherical triangles, but their use reduces by one or two the number of operations with tables.

1. The haversine formulas for an angle in terms of three sides, the s.s.s. case, and the haversine formulas for a side in terms of three angles, the a.a.a. case:

a. $hav\ \mathrm{A} = \sin(\mathrm{s} - \mathrm{c}) \sin(\mathrm{s} - \mathrm{b}) \csc \mathrm{b} \csc \mathrm{c}$; *and so forth.*
b. $hav\ \mathrm{a} = -\cos \mathrm{S} \cos(\mathrm{S} - \mathrm{A}) \csc \mathrm{B} \csc \mathrm{C}$; *and so forth.*

The justification of the above formulas follows immediately from the theorem given above — based on the definition of the haversine function — and the expressions derived in section 9 for $\sin \frac{1}{2} A$ and $\sin \frac{1}{2} a$, respectively.

The use of these haversine formulas rather than the half-angle formulas, from which they are an immediate consequence, saves taking a square root and multiplying the half angle by two.

2. The haversine formulas for a side in terms of the other two sides and the included angle, the s.a.s. case, and the haversine formulas for an angle in terms of the other two angles and the included side, the a.s.a. case:

a. $hav\ \mathrm{a} = hav\ (\mathrm{b} - \mathrm{c}) + \sin \mathrm{b} \sin \mathrm{c}\ hav\ \mathrm{A}$; *and so forth.*
b. $hav\ (\pi - \mathrm{A}) = hav\ (\mathrm{B} + \mathrm{C}) - \sin \mathrm{B} \sin \mathrm{C}\ hav\ \mathrm{a}$; *and so forth.*

Since the haversine is defined in terms of the cosine, the cosine laws are obvious points of departure in deriving the haversine laws. For formula (a) replace the cosines of the side and opposite angle in the law of cosines for sides by their equivalent expressions in terms of haversines.

$$\begin{aligned}
\cos a &= \cos b \cos c + \sin b \sin c \cos A \\
1 - 2 \operatorname{hav} a &= \cos b \cos c + \sin b \sin c\,(1 - 2 \operatorname{hav} A) \\
1 - 2 \operatorname{hav} a &= \cos b \cos c + \sin b \sin c - 2 \sin b \sin c \operatorname{hav} A \\
1 - 2 \operatorname{hav} a &= \cos (b - c) - 2 \sin b \sin c \operatorname{hav} A \\
1 - 2 \operatorname{hav} a &= 1 - 2 \operatorname{hav} (b - c) - 2 \sin b \sin c \operatorname{hav} A \\
\operatorname{hav} a &= \operatorname{hav} (b - c) + \sin b \sin c \operatorname{hav} A
\end{aligned}$$

* See Bowditch's *Practical Navigator* or *Useful Tables*, published by the Hydrographic Office (Washington, D.C.: Government Printing Office).

In this case polarization of (a) does not help us to derive (b). To complete the derivation up to the last step, paraphrase the right-hand side of the derivation given above, beginning with the law of cosines for angles. Then use a familiar property of the cosine.

$$\begin{aligned}
\cos A &= -\cos B \cos C + \sin B \sin C \cos a \\
\cos A &= -\cos B \cos C + \sin B \sin C\,(1 - 2 \operatorname{hav} a) \\
\cos A &= -\cos B \cos C + \sin B \sin C - 2 \sin B \sin C \operatorname{hav} a \\
\cos A &= -\cos (B + C) - 2 \sin B \sin C \operatorname{hav} a \\
1 + \cos A &= 1 - \cos (B + C) - 2 \sin B \sin C \operatorname{hav} a \\
1 + \cos A &= 2 \operatorname{hav} (B + C) - 2 \sin B \sin C \operatorname{hav} a \\
1 - \cos (\pi - A) &= 2 \operatorname{hav} (B + C) - 2 \sin B \sin C \operatorname{hav} a \\
2 \operatorname{hav} (\pi - A) &= 2 \operatorname{hav} (B + C) - 2 \sin B \sin C \operatorname{hav} a \\
\operatorname{hav} (\pi - A) &= \operatorname{hav} (B + C) - \sin B \sin C \operatorname{hav} a
\end{aligned}$$

The haversine laws involve looking up one less logarithm than the respective cosine laws from which they are derived.

See example 3 on page 201 for applications of the haversine laws to particular triangles.

12. Illustrative Examples

EXAMPLE 1. *By means of the law of sines and the law of cosines for sides solve the triangle two of whose sides are respectively* 110° 30′ *and* 65° 15′ *and whose included angle is* 125° 20′. *Use a slide rule.*

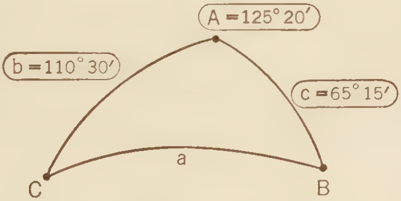


FIGURE 209

Figure 209 shows the labeling of the given parts. The cosine law will evaluate the third side, a, after which the sine law may be applied to compute the two unknown angles, B and C.

$$\begin{aligned}
\cos a &= \cos 110^\circ 30' \ \cos 65^\circ 15' + \sin 110^\circ 30' \ \sin 65^\circ 15' \ \cos 125^\circ 20' \\
&= -0.1465 - 0.490 \\
&= -0.637 \\
a &= 129^\circ 40' \\
\sin B &= \sin 110^\circ 30' \ \sin 125^\circ 20' \ \csc 129^\circ 40' \\
&= 0.994 \\
B &= 84^\circ \text{ or } 96^\circ \\
\sin C &= \sin 65^\circ 15' \ \sin 125^\circ 20' \ \csc 129^\circ 40' \\
&= 0.964 \\
C &= 74^\circ 30' \text{ or } 105^\circ 30'
\end{aligned}$$

When side a is evaluated the order of magnitude of the sides is seen to be $c < b < a$. Hence, the order of magnitude of the angles is $C < B < A$. Since the second quadrant angle C, 105° 30′, is larger than either of the possibilities for angle B, angle C is obviously 74° 30′. However, both answers offered for angle B satisfy the law of magnitude of the parts of the triangle. Hence, the proper quadrant of angle B must be determined by investigating the sign of its cosine in the law of cosines for sides.

$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$

$$\cos B = \frac{\cos 110^\circ 30' - \cos 129^\circ 40' \cos 65^\circ 15'}{\text{(a positive quantity)}} = \frac{-0.350 + 0.268}{(+)}$$

$\cos B = (-)$. Therefore B is second quadrant and $= 96°$.

Therefore, $\boxed{a = 129^\circ 40';\ B = 96^\circ;\ C = 74^\circ 30'.}$

EXAMPLE 2: *Solve completely the spherical triangle whose three sides are respectively* 38° 05′ 26″, 116° 22′ 07″, *and* 151° 38′ 43″. *Use the half-angle formulas and logarithms.*

Figure 210 shows the labeling of the given parts.

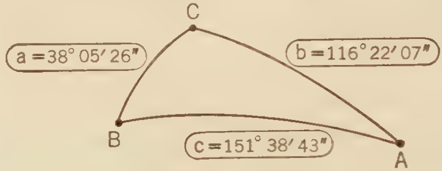


FIGURE 210

$$\tan \tfrac{1}{2} A = \frac{r}{\sin (s - a)}, \quad \tan \tfrac{1}{2} B = \frac{r}{\sin (s - b)}, \quad \tan \tfrac{1}{2} C = \frac{r}{\sin (s - c)},$$

where $s = \frac{1}{2}(a + b + c)$ and $r = \sqrt{\dfrac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}$

$a =$	38° 05′ 26″				
$b =$	116° 22′ 07″				
$c =$	151° 38′ 43″				
$2s =$	306° 06′ 16″				
$s =$	153° 03′ 08″	l csc 10.34373			
$s - a =$	114° 57′ 42″	l sin 9.95741	l csc 10.04259		
$s - b =$	36° 41′ 01″	l sin 9.77626		l csc 10.22374	
$s - c =$	1° 24′ 25″	l sin 8.39010			l csc 11.60990
	($s =$ 153° 03′ 08″)				
r		2 log 8.46750	log 9.23375	log 9.23375	log 9.23375
$\frac{1}{2}A =$	10° 41′ 58″		l tan 9.27634		
$\frac{1}{2}B =$	15° 59′ 58″			l tan 9.45749	
$\frac{1}{2}C =$	81° 50′ 37″				l tan 10.84365

$A =$	21° 23′ 56″
$B =$	31° 59′ 56″
$C =$	163° 41′ 14″

Features to be noted:

1. The arithmetic involved in finding $s - a$, $s - b$, $s - c$ should be checked by adding to give back s as is shown in parentheses.

2. Although r need never be explicitly evaluated, the line on which its logarithms are written, should be labeled r.

EXAMPLE 3: *Given the spherical triangle in which two sides and the included angle are respectively* 83° 54′ 18″, 22° 38′ 25″, *and* 79° 37′ 47″, *find the third side by the haversine law and the two angles by the sine law.* (*This type of solution is sometimes referred to as the "haversine-sine" method.*)

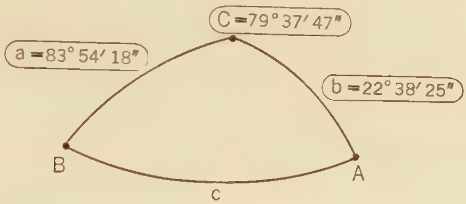


FIGURE 211

The labeling of the given parts is indicated in Figure 211. Then,

$\underline{\text{hav } c = \text{hav } (a - b) + \sin a \sin b \text{ hav } C}$

$\underline{\sin A = \sin C \sin a \csc c}$

$\underline{\sin B = \sin C \sin b \csc c}$

a = 83° 54′ 18″	l sin 9.99753		l sin 9.99753	
b = 22° 38′ 25″	l sin 9.58540			l sin 9.58540
$a - b$ = 61° 15′ 53″		n hav 0.25962		
C = 79° 37′ 47″	l hav 9.61278		l sin 9.99285	l sin 9.99285
	l hav 9.19571	n hav 0.15693		
c = 80° 23′ 34″		n hav 0.41655	l csc 10.00614	l csc 10.00614
A = { 82° 45′ 36″ / 97° 14′ 36″ }			l sin 9.99652	
B = 22° 35′ 06″				l sin 9.58439

The law of magnitude of parts of a triangle determines that $B = 22°$ and not 158° but fails to select the proper A. Testing the quadrant of angle A by a law of cosines with a slide rule will accomplish this:

$$\cos A = \frac{\cos a - \cos b \cos c}{(+)} = \frac{\cos 84° - \cos 23° \cos 80°}{(+)}$$

$$= \frac{0.1045 - (0.920)\,(0.1736)}{(+)} = \frac{0.1045 - 0.1595}{(+)} = \frac{(-)}{(+)} = (-)$$

Therefore A is second quadrant. Hence,

$$\boxed{c = 80°\,23'\,34''; \; A = 97°\,14'\,36''; \; B = 22°\,35'\,06''}$$

Features to be noted:

1. All formulas to be used were stated in computation form and underscored at the outset.

2. Since hav c involves a sum, the antilog of 9.19571 (that is, 0.15693) in column two must be found from tables. The labeling given 9.19571 could have been log. instead of log haversine.

3. In many (perhaps most) cases the law of magnitude of parts of a triangle will determine the quadrant of both unknown angles. This more troublesome and frequently occurring case is given here to provide a complete illustration of the haversine-sine method.

4. Both tentative values of A were read from the tables, which accounts for the slight discrepancy in their supplementary relationship.

EXAMPLE 4: *By logarithms and using only Napier's analogies solve the spherical triangle two of whose sides are respectively* 122° 39′ 48″ and 44° 13′ 05″ *and whose included angle is* 72° 01′ 21″.

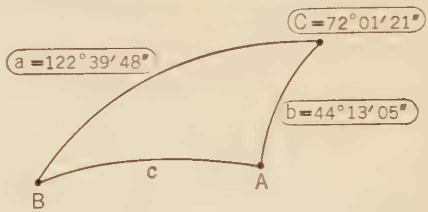


FIGURE 212

The labeling of the given parts is shown in Figure 212.

$$\tan \tfrac{1}{2}(A - B) = \cot \tfrac{1}{2} C \sin \tfrac{1}{2}(a - b) \csc \tfrac{1}{2}(a + b)$$

$$\tan \tfrac{1}{2}(A + B) = \cot \tfrac{1}{2} C \cos \tfrac{1}{2}(a - b) \sec \tfrac{1}{2}(a + b)$$

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2}(a - b) \sin \tfrac{1}{2}(A + B) \csc \tfrac{1}{2}(A - B)$$

a	= 122° 39′ 48″			
b	= 44° 13′ 05″			
$a+b$	= 166° 52′ 53″			
$a-b$	= 78° 26′ 43″			
C	= 72° 01′ 21″			
$\frac{1}{2}C$	= 36° 00′ 40″	l cot 10.13856	l cot 10.13856	
$\frac{1}{2}(a-b)$	= 39° 13′ 22″	l sin 9.80095	l cos 9.88913	l tan 9.91181
$\frac{1}{2}(a+b)$	= 83° 26′ 26″	l csc 10.00285	l sec 10.94221	
$\frac{1}{2}(A-B)$	= 41° 12′ 33″	l tan 9.94236		l csc 10.18124
$\frac{1}{2}(A+B)$	= 83° 52′ 58″ *		l tan 10.96990	l sin 9.99752
$\frac{1}{2}c$	= 50° 55′ 54″			l tan 10.09057

c	= 101° 51′ 48″
A	= 125° 05′ 31″
B	= 42° 40′ 25″

* Interpolation here is from the nearer minute — contrary to general usage which is from the *smaller* minute. The divergence from general custom here is due to the large tabular difference.

13. Problems for Appendix II

1. Solve by the half-angle formulas the spherical triangle in which $a = 38° 05' 26''$, $b = 116° 22' 07''$, and $c = 151° 38' 43''$.

2. Solve the spherical triangle ABC in which $A = 63° 42' 18''$, $b = 123° 15' 20''$, and $c = 27° 04' 52''$, by Napier's analogies.

3. Find the distance and the initial course for a great-circle track from San Francisco (lat. 37° 47′ 30″ N., long. 122° 27′ 49″ W.) to Sydney (lat. 33° 51′ 41″ S., long. 151° 12′ 39″ E.). Use the haversine formula *2 a* and the law of sines.

4. A man flies in a generally westward direction from a place A (lat. 30° 25′ N., long. 70° 20′ W.) to a place B (lat. 20° 15′ S.) a distance of 4232.5 nautical miles. Find the longitude of B, using haversine formula *2 a*.

5. A man flies on a great-circle course in a generally westward direction from a place A (lat. 25° 44′ 40″ N., long. 30° 16′ 25″ W.) to a place B for which the longitude is 70° 44′ 15″ W. The angle of departure (PAB) is 25° 35′ 40″. In the spherical triangle ABP (P is the North Pole) solve for the distances AB and PB by Napier's analogies.

6. Solve the spherical triangle ABC in which $a = 40° 05' 26''$, $A = 29° 42' 34''$, and $c = 26° 21' 18''$, by the law of sines and Napier's analogies.

7. The sides of a spherical triangle are $a = 50° 12' 04''$, $b = 116° 44' 48''$, and $c = 129° 11' 42''$. Find the angle C by using the haversine formula (*1 a*).

8. Solve the spherical triangle in which $C = 82° 33' 31''$, $a = 99° 40' 48''$, and $B = 114° 26' 50''$ by the law of cosines for angles and the law of sines.

9. Given $a = 122° 37' 14''$, $b = 88° 12' 39''$, and $c = 43° 58' 07''$, find angle B by a haversine law and angle A by the sine law. Determine in which quadrant A lies.

10. Using a law of cosines, find by logarithms the side a of the spherical triangle in which $b = 47° 44' 00''$, $c = 53° 19' 28''$, and $A = 52° 30' 00''$.

11. Given $a = 80° 10' 00''$, $b = 104° 25' 00''$, and $c = 139° 40' 32''$, find B by the haversine formula

$$\text{hav } B = \sin (s - a) \sin (s - c) \csc a \csc c.$$

12. Solve the spherical triangle by the tangent of the half-angle formulas.

$$a = 44° 25' 13'', \quad b = 67° 09' 14'', \quad c = 91° 32' 15''.$$

13. A pilot flies on a great-circle course with average speed 190 nautical miles an hour. He starts in the northern hemisphere on course 52° 13′ 20″ and, after changing his longitude by 115°, he is on course 158° 17′ 45″. How long has this flight taken the pilot, by how much did he change his latitude, and how close did he come to the North Pole?

14. Given $d = 38° 17'$ S., lat. $= 24° 33' 30''$ N., $t = 28° 27' 38''$, find h by the haversine formula

$$\text{hav } (90° - h) = \text{hav (lat.} - d) + \cos \text{lat.} \cos d \text{ hav } t.$$

15. If a ship sails along the arc of a great circle from Point Loma (lat. 32° 39′ 48″ N., long. 117° 14′ 37″ W.), outside San Diego Harbor, to Otago Harbor, Taivoa Head Light (lat. 45° 46′ 55″ S., long. 170° 44′ 02″ E.), the harbor at

Dunedin, New Zealand, what initial course should she set and how far does she sail? Solve completely by Napier's analogies and check by the law of sines.

16. A navigator estimates the azimuth of a star, of declination 42° 10′ N., to be 65° 30′ east of north at the instant the assistant navigator observes the altitude of the star to be 36° 28′ 30″. If these observations were made at a chronometer instant indicating that the Greenwich hour angle of the star was 250° 03′ 48″, find, by the methods of Appendix II, where the point of observation *might* have been. If the observations were made aboard a ship, where *must* the ship have been?

Instruments for Observing Spherical Trigonometry Data

PART ONE: *The Sextant*

14. General Description of the Sextant

The sextant is a relatively simple instrument by which the angle at the point of observation between the lines of sight to two points can be measured. Chief of such observations is that of finding altitudes of celestial bodies. The principle involved in this procedure is described below.

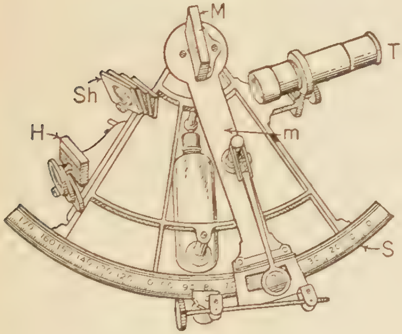


FIGURE 213

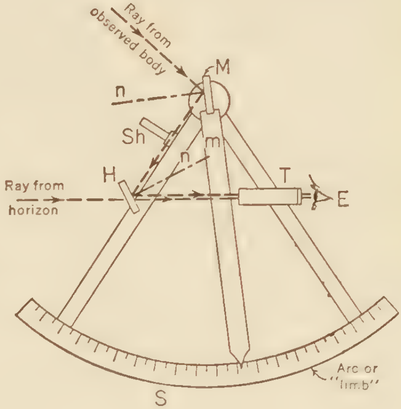


FIGURE 214

Figure 213 shows a modern sextant. Figure 214 is a simplified sketch of the principal parts of this instrument in the vertical position necessary for observing altitudes. M is a mirror, the "index glass," perpendicular to the plane of the arc or "limb" of the sextant, and mounted on the arm m. This arm is pivoted at the top at the center of the arc of the sextant. The amount of turning of the index glass M is indicated by the position of the pointer on the lower end of the arm on the scale S of the arc or limb of the sextant. H is called the horizon glass. It too is perpendicular to the plane of the arc or limb, but it is fixed to the frame. The half of the horizon glass nearer the frame is silvered to form a mirror; the other half is clear. T is a telescope trained on the center of the horizon glass. The index mirror is in the field of vision reflected from the horizon glass.

15. Observing the Altitude of the Sun

Suppose the altitude of the sun at sea is required. The observer faces the sun, holds the sextant vertically with his right hand, places an eye, E, to the telescope T (see Figure 214), and so tilts the sextant in the vertical plane as to see the horizon through the clear half of the horizon glass. Because of the fixed position of this horizon mirror described above, the observer, by means of the silvered half of the horizon glass, can, at the same time, look into the movable mirror for all positions of this movable mirror. If the movable mirror be turned, by rotating the arm about its pivot, the image of the sun can be reflected from the movable mirror onto the silvered half of the horizon mirror and into the telescope where it can be seen. Before this is done, however, a shade of colored glass, Sh, must be turned down between the two mirrors to lessen the glare of the sun.

When the observer has moved the arm so that he can simultaneously see the horizon through the clear half and the reflection of the sun in the silvered half of the horizon glass, the pointer on the arm will indicate the sun's altitude on the scale. This is because there is a constant relation between the angle that is formed by the two mirrors (which is evidently proportional to the pointer reading on the scale), and the angle that is formed by the ray incident on the movable mirror and the ray reflected from the fixed mirror. The particular constant relation here is that the former angle is half the latter, as may be seen by the

THEOREM: *If a ray of light in one plane is reflected in succession by two mirrors, the angle through which the ray is thereby turned will be twice the angle between the planes of the mirrors.*

In Figure 215 the light-ray from the sun, Σ, is shown reflected by the movable mirror, M, onto the silvered half of the

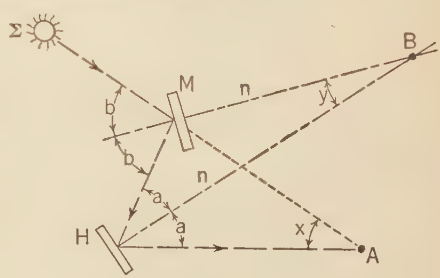


FIGURE 215

horizon glass, H. The angle at A is then the angle through which the sun's ray has been turned by reflection from the two mirrors. The angle at B, formed by the normals to the two mirrors, is equal to the angle between the two mirrors. By use of the theorem that "an exterior angle of a plane triangle equals the sum of the two remote interior angles" and the reflection law that "the angle of reflection equals the angle of incidence":

$$2b = 2a + x \text{ and } b = a + y$$
$$\text{Therefore, } 2b = 2a + 2y = 2b - x + 2y$$
$$\text{Hence, } x = 2y, \ q.e.d.$$

By sighting the horizon through the clear half of the horizon glass while reflecting the sun's image into the telescope, the angle between the originally incident and finally reflected ray is made to be precisely the altitude of the sun. The scale is designed to read twice the angle of inclination of the two mirrors and hence exactly the sun's altitude.

16. Observing Altitudes of Fixed Stars

The sextant is also used to observe the altitude of fixed stars. This must be done immediately after sunset or immediately before sunrise when both stars and horizon are visible. The procedure here differs from that for the sun in but one particular: The order of sighting horizon and observed body is reversed. Because of the conspicuousness of the sun, the horizon can first be sighted and the sun's reflection then picked up in the movable mirror. This order in the case of a fixed star would be difficult because of the many stars and their closeness together. With the pointer set to zero on the scale the telescope should first be pointed at the star. The reflected image of the star will then actually coincide with the image of the star as seen through the horizon glass. Then the sextant should be slowly tilted in the vertical plane toward the horizon while the arm is so moved that the reflected image of the star remains in the mirror half of the horizon glass until the horizon is visible through the clear half of the horizon glass.

17. The Simplicity of the Sextant

The simplicity of the sextant admirably adapts it for use aboard ship where a stable level platform is at best impractical. No spirit levels need be adjusted by level screws. The sextant can readily be fixed in proper adjustment, as follows. The mirrors must be perpendicular to the plane of the limb and the axis of the telescope parallel to this plane. Then all that is necessary is that the instrument be held in the vertical plane of the observed body. This can readily be accomplished by tilting the sextant slightly from side to side with the result that the reflection of the observed body will appear to move in a circular arc. When this image is at its lowest point the sextant is in the vertical plane of the observed body. That this is so is due to the fact that a reading obtained when the sextant is not vertical will always be larger than the correct one as is suggested by Figure 216.

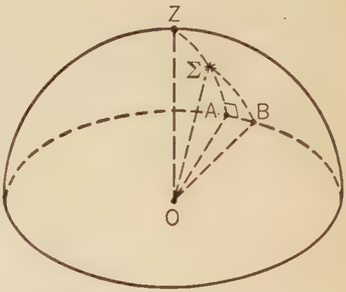


FIGURE 216

Sextant observations are usually accurate to within 10″ of arc.

The sextant takes its name from the fact that the scale on which the altitude is read, the limb, was originally an arc of 60° or one-sixth of a circle. Since the graduations on this scale read twice the actual arc, such a sextant will read angles up to 120° or will allow altitudes to be measured from the point on the observer's horizon on the other side of the observer's zenith from the celestial body. Actually present-day sextants read up to nearly 180°, which means that the arc of the limb is nearly 90°. An *octant* is a "sextant" for which the limb is one-eighth of a circle and therefore reads altitudes up to 90°.

18. Plane Trigonometry Uses of the Sextant

The sextant can also be used to measure the approximate angle subtended at the observer by any two objects such as headlands, lights, and so forth. Such observations give data for plane rather than spherical triangle solution. The sextant must be held in the plane determined by the lines of sight to the two objects. Then, while sighting on one object, through the clear half of the horizon glass, the observer moves the arm so that the image of the second object is brought into coincidence with the first object in the mirror half of the glass.

19. The Use of an Artificial Horizon

Frequently when observations must be made, the horizon is obscured by poor visibility or by an island or other land in the direction of sighting. In this case an artificial horizon is used. This usually consists of a dish of mercury shielded from the wind by a glass roof and placed in front of the observer in the vertical plane of the body to be observed. The mercury acts as a level mirror. With the sextant vertical the observer sights through the telescope on the artificial horizon and moves nearer to or further away from this artificial horizon until he sees the image of the body to be observed reflected from the surface of the mercury into the telescope. By rotating the arm, the movable mirror can be tilted, as before, so as to reflect the body's image onto the silvered half of the horizon glass and from there into the telescope. If the observed body is the sun, both images must be shielded. Different colored glasses are generally used to give two images of the body of different colors. When these two images appear coincident in the telescope — one reflected directly from the mercury through the clear half of the horizon glass, and the other reflected from the movable mirror onto the silvered half of the horizon glass and then into the telescope — the pointer reading on the scale is *double* the altitude, as Figure 217 will

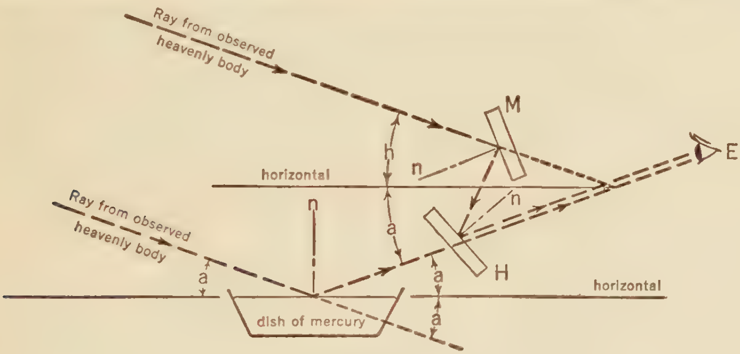


FIGURE 217

show. The angle marked h, between the observed ray and the horizontal, is the altitude, by definition. Because of the immense distance of the heavenly body from the point of observation, the two rays from the body can be considered parallel and hence as making equal angles with the horizontal. Therefore,

$$\text{angle } h = \text{angle } a.$$

Therefore, the angle between the ray incident on the movable mirror and the ray reflected from the horizon glass is double the altitude. But the description given above for the real horizon case shows that the pointer reading gives the angle between the initial and finally reflected ray. *Therefore in the case of the artificial horizon the pointer reading is double the altitude.*

20. The Bubble Sextant or Octant

For altitude observations in an airplane the ordinary sextant is of limited use. The airplane is frequently above clouds which totally obscure the earth. Whenever the horizon is visible, corrections for dip are large and depend on accurate information as to the height of the plane. Furthermore, since plane flights are relatively short, it may be necessary to take sights on stars at some time in the night other than at sunrise and sunset when the horizon and the stars are both visible. Accordingly, for airplane observations the sextant is replaced by the ***bubble octant,*** in which the horizon glass is replaced by a leveling bubble.

Although the fundamental principles involved in all bubble octants are much the same — and are much the same as for the marine sextant, with the exception of the additional feature which permits the substitution of a leveling bubble for the horizon — the various makes and models of bubble octants differ markedly in external appearance. Furthermore, in contrast to the marine sextant, the essential working parts of bubble octants are so enclosed in casings as to conceal the principles on which the instruments operate. Accordingly, instead of a picture of a bubble octant, it will be enough here (Figure 218) to present merely a diagram of the essential, but hidden, parts of one standard model of the bubble octant.

When the eye is placed at the eyepiece, E_1, the horizon is brought into the field of vision by simply pointing the lens, L, to the horizon. The image of the horizon is then reflected downward by a prism, located at P_1 behind this lens, and then upward through the eyepiece by another prism, located at P_2. When the horizon is being observed, these two reflecting prisms constitute the counterpart of the horizon glass in the marine sextant.

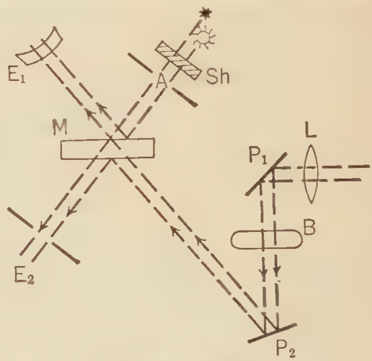


FIGURE 218

The index glass is a "transparent mirror," located at M in the line of sight from the eyepiece E_1 to the second reflecting prism, P_2. This mirror consists of a piece of clear glass through which objects can be seen as through a windowpane and from either surface of which objects can be seen by reflection when the glass is properly inclined. This index mirror rotates about its horizontal axis perpendicular to the line of sight between E_1 and P_2. The rotation of the index mirror is controlled by a knurled wheel on the outside casing of the instrument, and the amount of this rotation is indicated on a dial.

When the altitude of the sun is being observed, the image of the horizon being used, the rays of the sun enter the opening, A, strike the upper surface of the index mirror, and are reflected into the eye at E_1. To reduce the sun's glare, a colored glass shade, *Sh*, is moved so as to be perpendicular to the rays entering *A*. The altitude of the sun is then measured by the amount of rotation of the index mirror (as read on the dial) necessary to make the sun's rays, which are reflected from the upper surface of the index mirror, parallel to the horizon rays, which are reflected by the two prisms through the index glass.

Now, in place of the image of the horizon the image of the leveling bubble in the bubble chamber, located at *B*, can be used. This bubble, imprisoned in a glass cell, lies in the line of sight between the two reflecting prisms, P_1 and P_2. It is illuminated in daylight by light entering the lens, *L*, and at night by a small light operated by a battery concealed in the handle of the instrument. The sun's altitude — or the altitude of any other observed heavenly body — is correctly read when the image of the sun — or any other observed body — appears in the exact center of this bubble when the bubble is free from the sides of its chamber. The size of this bubble can be regulated by rotating a knob on the outside of the instrument.

The system of observing altitudes which is described above — in which the eye is placed at E_1, the bubble or horizon viewed through the transparent index mirror, and the image of the observed body viewed as reflected from the upper surface of the index glass — is used when the observed body is particularly conspicuous, as is the sun in the daytime and the moon at night. For stars the system is reversed: The eye is placed at E_2, the star observed directly through the transparent index glass, and the image of the bubble is observed as reflected from the under side of the index glass.

Bubble octants can be used at any time during day or night. Unfortunately, largely because of the effect of the acceleration of an airplane on the bubble, altitude observations made with a bubble octant in an airplane are, at present, even under the best of conditions, accurate to within but three to five minutes of arc.

PART TWO: *The Chronometer*

21. General Description

The chronometer is an exceptionally accurate clock, protected and handled with great care. It is used aboard ship for keeping Greenwich civil time, from which hour angles can be computed on the basis of certain celestial observations and the *Nautical Almanac*.

The chronometer is set to Greenwich civil time at an observatory and, regardless of the amount of subsequent error, is not again set to this time until three or four years later when it is returned to the observatory for cleaning and resetting. Despite the care in constructing and handling a chronometer, it will practically always have a *rate of error* which results, at any given time, in an *accumulated error*. The usefulness of a chronometer depends not upon the size of its accumulated error nor on the size of its rate of error but upon the uniformity of this rate. The amount of error is frequently checked by radio time signals and the results carefully recorded. From these records the rate of error is calculated for the interval between each pair of consecutive recordings. On the basis of these rates the rate of error at any present time is assumed. From the amount of error at the most recent radio checking and the assumed rate of error since that time, the Greenwich time at any desired instant can be computed. This is usually done to tenths of seconds of time. Chronometers, in specially constructed boxes from which they are never removed except to be repaired, can frequently be seen in jewelers' and watchmakers' windows.

Because of the importance and delicacy of chronometers they are kept at all times in the same protected location aboard ship. When observations and computations involving time are to be made, a watch is very carefully compared with a chronometer just before (or just after) observations and then the observations are timed by the watch. Great care is exercised in insuring that the chronometers never run down. They are designed to run for fifty-six hours without rewinding, and they are wound at the same time every day by the same person who reports the winding to the ship's master on each occasion. On large ships three chronometers are carried. The one with the most constant rate of error is termed the *standard chronometer* to which the others are compared.

On small ships and in airplanes a *chronometer watch* is used. This is a large watch of accurate performance, protected and cared for as are chronometers.

PART THREE: *The Azimuth Circle and Compass*

22. Bearings and Azimuth

The azimuth circle is an attachment to a compass by which the bearings of terrestrial objects (buoys, lighthouses, ships, and so forth) may be observed and by which the azimuths of celestial bodies may be approximated. Plane sailing, which requires only plane trigonometry, employs the concept of bearing; azimuth is a spherical trigonometry concept necessary in great-circle sailing. Since azimuth can logically be considered a particular case — that of a celestial body — of the more general concept of bearings, the more general concept will be considered first.

Definition: *The **bearing of an object** M **from a point** O is the direction of* M *from* O *when* M *is in the horizontal plane of* O, *and is the projection of this direction onto the horizontal plane of* O *when* M *is not in this plane.*

This direction is necessarily described with reference to a standard horizontal direction and is the angle at O between the standard direction and either the direction of M, if horizontal, or its horizontal projection, if M is not horizontal.

Definition: *When the standard direction is that of the northern part of the meridian through* O *and when the sense of measuring the angle from the standard direction is clockwise, the bearing is termed a **true bearing**.*

To say that the bearing from a ship of a lighthouse is 282° true is to say it lies 12° north (geographic, not magnetic, north) of west of the ship.

Definition: ***Compass bearings** are bearings in which the standard direction is the north direction indicated by a magnetic compass.*

This direction will seldom be exactly a true bearing when a magnetic compass is used, since the magnetic north pole is over a thousand miles from the north geographic pole. A compass bearing, like a true bearing, is read clockwise as so many degrees from the indicated north direction.

Definition: ***Relative bearings** are bearings referred to the direction of motion of a ship, that is, its "head."*

If the ship in the illustration above is sailing northeast, the bearing of the lighthouse relative to this head would be 123° westward. If the compass bearing of the lighthouse were 72° west, the compass needle would be pointing 6° west of true north.

Comparison of azimuth of celestial bodies with the concept of true bearing will show that the azimuth of a celestial body at a given point and its true bearing at this point are identical. The terms azimuth and bearing of celestial bodies are, accordingly, often used interchangeably.

23. Compass Cards

The azimuth or bearing of a celestial body can theoretically be observed by sighting its line of direction from the point of observation over a compass. The angle at the center of the horizontal compass card between true north, as is indicated by this compass card, and the horizontal projection of the line of sight would be the azimuth. The particular compass card used may be the card of a ***pelorus*** or "dummy compass," which is merely a card set by hand to point north by reference to some actuated compass card, or the card of either a ***magnetic compass*** or a ***repeating compass of a gyro compass.*** The mechanics of these fundamental types of compasses are briefly described below. For the present, however, we shall assume that some sort of a circular compass card is given which has the following characteristics:

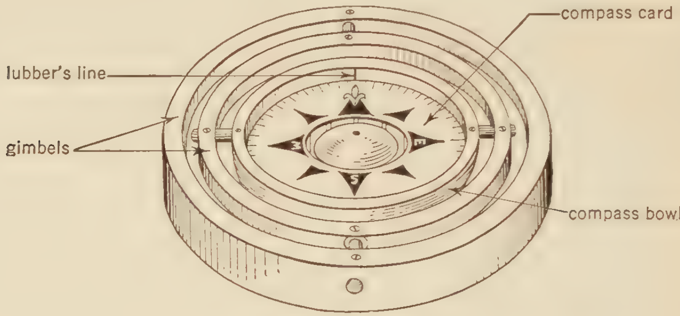


FIGURE 219

1. The rim is divided into 360° of arc with the principal ***points of the compass*** marked, that is, north at 0°, east at 90°, southwest at 225°, and so forth.

2. The card is pivoted at its center and forced — by a setting by hand in a pelorus, or by magnets in a magnetic compass, or by electrical connections with the master gyro compass below decks in a repeating compass — to point true north with known corrections.

3. The card is mounted on top of a bowl in which the card is free to turn on its pivot to point north. The bowl is suspended in ***gimbels*** so that the attraction of gravity will keep the card horizontal despite the roll and pitch of the ship. The gimbels consist of two concentric rings around the outside of the top of the compass bowl. The outer ring is fixed to the ship by a standard. The inner ring is pivoted to the outer ring at opposite ends of a diameter, and the compass bowl is pivoted to the inner ring at opposite ends of the diameter perpendicular to the diameter of the pivots between the two rings. (See Figure 219.) In the case of the pelorus these gimbels allow the card to be tilted out of the horizontal in any direction if this is necessary for sighting on objects markedly out of the horizontal plane of the card.

4. On the inside of the bowl just above the compass card is a mark indicating the direction of the "ship's head." This mark is called the ***lubber's line*** of the compass. The lubber's line marks that diameter of the bowl which is parallel to the keel of the ship. Hence, the compass

reading at the lubber's line gives the "ship's head" or compass direction of motion of the ship.

24. Observing Bearings of Terrestrial Objects

The azimuth circle, a diagram of which is given in Figure 220, is an attachment to a compass. It consists of a ring, *R*, which is graduated to 360° running counterclockwise (unlike the compass) and has two pairs of sighting vanes erected in pairs at diametrically opposite points. The azimuth circle fits over the outer edge of the compass and is free to be turned around the compass card concentrically with this card.

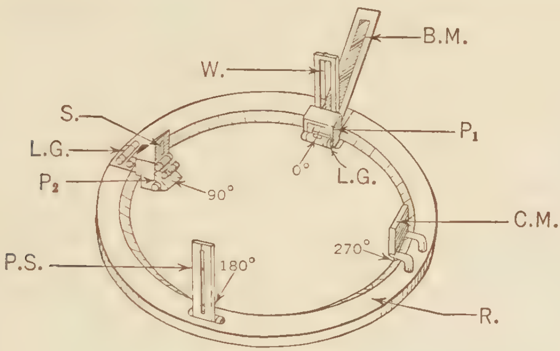


FIGURE 220

Directly above the 0° mark on the azimuth circle is a vertical wire, *W*, in a frame. Diametrically opposite on the 180° mark is a peep sight, *P.S.* To obtain the bearing of an object, an observer turns the azimuth circle around so that by sighting through the peep sight he can see the object on the vertical wire of the opposite vane. At the base of the vertical wire is a right-angled reflecting prism, P_1, marked with a line agreeing with the vertical wire. This prism reflects the compass card onto the observer's field of vision, so that he sees both the observed object and the compass card at the same time. The position of the vertical wire on this reflected compass card is then the compass bearing of the observed object. By applying whatever correction is necessary to transform a compass reading to a reading based on true north, the object's true bearing is given. The reading on the azimuth circle opposite the lubber's line on the compass gives the bearing of the observed object relative to the ship's head — as so many degrees clockwise from the ship's head. It also gives the ship's head relative to the observed object — as so many degrees counterclockwise from the observed object.

Figure 221 illustrates the azimuth circle used for observing compass bearings of terrestrial objects.

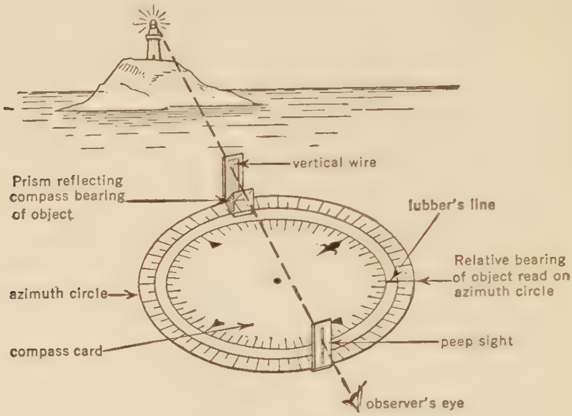


FIGURE 221

As both the vertical wire and the opposite peep sight are about an inch or so in height, it is possible to sight on distant objects which are considerably out of the horizontal plane of the compass card. Furthermore, the bowl can be tipped slightly in the gimbels about that horizontal axis of the compass card which is perpendicular to the vertical plane of sighting. In this way a bearing can be obtained on an object farther out of the horizontal plane of the compass card than the sighting vanes will accommodate. The azimuth circle is fitted with a level glass, *L.G.*, perpendicular to the line of sighting to check against tilting the azimuth circle about an axis in the plane of sighting.

25. Observing Azimuths of Celestial Bodies Other than the Sun

The azimuths of celestial bodies other than the sun are obtained as are the bearings of terrestrial bodies (described above) with the addition of reflections of the celestial bodies in a black mirror. This mirror, *B.M.*, is horizontally hinged at the base of the vane of the vertical wire and on the outside of this vane. The mirror is tilted on its horizontal axis to reflect the particular heavenly body in the black mirror behind the vane of the vertical wire. When the azimuth circle is so turned as to place this reflection directly behind the vertical wire, the compass reading of this wire (as seen by the lighted compass card reflected by the right-angled prism at the base of the vane) is the azimuth of the particular heavenly body. As only a few stars, all of them relatively bright, are generally

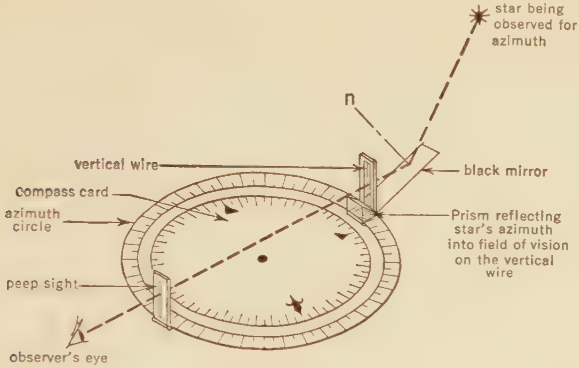


FIGURE 222

used in navigation, little trouble is encountered in reflecting the particular star which is to be observed. Figure 222 illustrates the azimuth circle as used to observe the azimuths of nocturnal celestial bodies.

26. Observing Azimuths of the Sun

For observing the sun's azimuth a separate pair of vanes is provided on the 90° and 270° diameter of the azimuth circle. One of these vanes is a concave mirror, *C.M.*, which focuses the sun's rays across the compass card into a narrow slit, *S*, in Figure 220, in the opposite vane. A reflecting prism, P_2, below this slit casts the sun's light down onto the compass card, making a bright streak of light on the compass card.

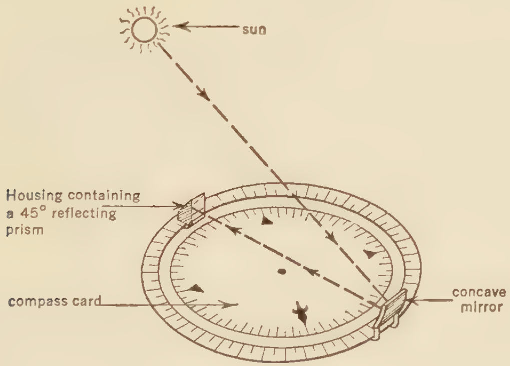


FIGURE 223

The reading at this streak of light is the sun's azimuth. Figure 223 illustrates the azimuth circle for observing the sun's azimuth.

27. Accuracy and Uses of Azimuth Observations

Since the compass card can be read to but a quarter of a degree at best, observed azimuths are only about one-sixtieth as accurate as observed altitudes. For this reason it is usual to observe altitudes of celestial bodies and compute their azimuths. Observed azimuths, however, are sufficiently accurate for slide rule computations.

In the problem of determining position by means of a fix from an estimated dead reckoning position (see page 155), the azimuths of the stars used in the fix are sometimes observed as an aid in their recognition and in checking the reasonableness of the dead reckoning position. However, on the chart the azimuth line (perpendicular to which the line of position is to be drawn at a distance in nautical miles from the dead reckoning position equal to the difference between the observed and computed altitudes) is the line of the *computed azimuth* of the observed star.

Observations of azimuth of celestial bodies from known points of observation are useful in determining the compass error. The azimuth of a star can be computed from the time of observation and from the position of the point of observation. Comparison of this with the star's azimuth as observed by the azimuth circle on the given compass to be checked will give the error. Such a procedure is consistent with the limitations in the accuracy of reading the azimuth circle, as this accuracy is precisely that of reading the instrument being checked, namely, the compass.

28. The Compass

The compass by itself is not an instrument which yields data for spherical trigonometry problems. As the previous sections have explained, however, an attachment to the compass, the azimuth circle, does provide spherical trigonometry data to a limited extent. Consequently, it seems in order to give a brief description of the principles which force the compass cards, which were mentioned in the description of the azimuth circle, to point north.

29. The Magnetic Compass

The earth appears to be a huge magnet with its north pole in northern Canada at about 70° north latitude and 97° west longitude — more than a thousand miles from the north geographic pole. Accordingly, freely suspended bar magnets, which are bars of iron that have been "magnetized" or made attractive to all other pieces of iron by having been placed in the electromagnetic field surrounding a coil of charged wire, place their north, or "north seeking," poles in the direction of this north magnetic pole. Consequently, by fixing bar magnets to the undersides of a compass card so that the magnets are parallel to the 0° and 180° diameter of the card, the attraction of the earth's magnetic pole on these magnets will make the compass card pivot on its axis so that the 0° mark on the card will point in the direction of the magnetic north. Usually a liquid which will not freeze at common temperatures is inserted into the compass bowl of the magnetic compass to partially float the compass card so that it will swing more freely on its pivot.

DEFINITION: *The **compass deviation** is the correction to be made on a given reading of a given magnetic compass to obtain the corresponding reading referred to magnetic north.*

This correction, due to the ship's magnetism, induced into the ship while it was being built (by lying for a long period of time in one position while its metal parts were hammered), is a property of the given compass in its given surroundings, and it is a function of the heading of the ship and the location of the ship.* This error can be largely compensated for by first testing the amounts of deviation for various headings and then experimentally placing various magnets and pieces of soft iron in the ***binnacle,*** or compass mounting, until the tabulated deviations largely disappear. Any remaining deviations must be available in tabular form so that they can be applied to compass readings to give magnetic readings.

DEFINITION: *The **compass variation** is the correction to be made on a given magnetic compass reading, previously corrected for deviation, to give the corresponding reading referred to true north.*

This correction, due to the magnetic poles' not being at the geographical poles, is a function of location on the earth's surface. On any given chart of a relatively small area the amount and direction of the compass variation for this region is shown.

30. The Gyro Compass. Precession of the Equinoxes.

When properly adjusted for large changes in the ship's speed and latitude, the gyro compass will point to the geographical or true north for all headings of the ship and for all positions within the limits of the latitude adjustment. Magnetic influences do not affect the gyro compass.

A gyroscope is essentially a body spinning rapidly about an axis of symmetry. The gyro compass makes use of the two physical principles characteristic of a gyroscope.

1. *The Stability Characteristic.* The axis of a gyroscope tends to remain fixed in direction in space unless acted on by a torque tending to

* Large changes in latitude may change a compass's deviation.

turn the axis out of the direction assumed when the body was first set in rotational motion.

2. *The Precessional Characteristic.* A torque tending to turn a gyroscope's axis of rotation about an axis perpendicular to the axis of rotation will not succeed in this but will cause the axis of rotation to turn about the axis perpendicular to the given axis of rotation and the axis of torque. The result will be that the axis of spin moves in the direction of the axis of torque. This motion of the axis of rotation is called *precession of the axis.*

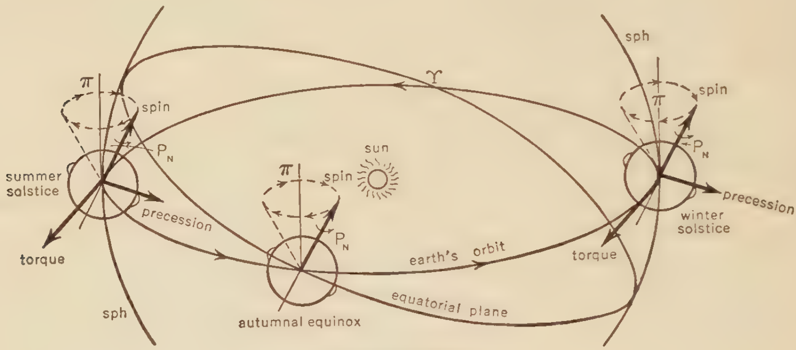


FIGURE 224

The earth itself is a gyroscope and therefore illustrates both these characteristics. Because of its considerable mass and rapid rotation on its axis, the direction of this axis tends to remain fixed in space — that is, in a direction at present closely approximated by the direction from the earth of Polaris. But because the earth is not a perfect sphere, since its equatorial diameters are larger than the polar diameter, the attraction on this equatorial bulge by the sun and the moon — the moon more than the sun — tends to turn the equatorial plane into the plane of the ecliptic. Figure 224 is a simplified sketch of the sun's part in this tendency.

Here the earth's orbit is represented as a circle with the sun as center. A sphere, *Sph,* is imagined with the sun as center and the earth's orbit as a great circle on it. On this sphere (*Sph*) is shown the projection from the celestial sphere of the intersection on this celestial sphere of the plane of the earth's equator. The earth is then shown, greatly enlarged and with an exaggerated equatorial bulge, in three positions in its orbit. For each of these positions the direction of the pole of the ecliptic is shown by a line marked Π.

The sun's attraction on the equatorial bulge is greater on the near side than on the far side, as attraction is inversely proportional to the squares of distances. Thus a torque is set up tending to turn the plane of the equator into the plane of the ecliptic. The magnitude of this torque is greatest at the solstices and zero at the equinoxes. Arrows, directed according to the right-hand screw convention, represent rotation and torque. Then the earth's axial rotation is shown by an arrow directed northward (the spin axis), and the torque is shown by an arrow directed out from the paper for all positions of the earth. As a result of the precessional characteristic of the gyroscope, instead of the torque's succeeding in turning the equatorial plane into the plane of the ecliptic, *the spin axis seeks the torque axis.* In Figure 224, therefore, the spin axis shown for the two solstices tends to tilt out from the paper. This tendency immediately makes the equatorial plane tilt so that the two solstice positions in Figure 224 would appear slightly farther back in their positions, in the orbit of the earth, shown at the extreme left and right in this figure. This means that the torque axis in Figure 224 would turn slightly to the left — in order to keep parallel to the line of the equinoxes and perpendicular to the line of the solstices. Since the spin axis seeks the torque axis, *the spin axis after first tilting out of the paper then tilts to the left.*

The continuous action of the above principles results in the spin axis of the earth (and therefore the polar axis of the celestial sphere) *precessing about the pole of the ecliptic*, much as the axis of a top — which is another gyroscope — with fixed foot precesses about the vertical through the foot. The polar axis moves as a generating element of a cone whose center is the earth's center, whose axis is the direction to the pole of the ecliptic, and whose semi-vertical angle remains about equal to the angle of the ecliptic. This changing of the direction in space of the polar axis does not materially change the angle of tilt of the earth's equator with respect to the ecliptic, but it does change the points of intersections on the celestial sphere of the ecliptic and the plane of the earth's equator. These points, the equinoxes, are thereby moved *backward*, or to the west (that is, contrary to the direction of the earth's orbital revolution). This gyroscopic phenomena of the earth, which is due to its equatorial bulge, is called the ***precession of the equinoxes.***

Because of the relatively inconsequential size of the equatorial bulge, the torque causing precession is very small. The rate of precession is accordingly very small, amounting to about fifty seconds of arc a year. A complete cycle requires about 26,000 years. In 3000 B.C. the star α Draconis was a good pole star. In A.D. 13,000 the star Vega will be a fairly good pole star. The first point of Aries takes its name from the constellation Aries, where this point was located some thousand years ago. Due to the precession of the equinoxes the first point of Aries is now in the constellation Pisces.

Figure 224 shows only what the sun does in causing precession of the equinoxes. About one third of the precession of the equinoxes is due to the sun and two thirds to the moon. The moon moves approximately in a plane inclined at about 5° with the ecliptic. Consequently, the effect of the attraction of the moon on the equatorial bulge of the earth is generally like that of the sun in producing a torque tending to turn the equator into the ecliptic. However, since the moon, unlike the sun, does not lie exactly in the ecliptic, the attraction of the moon on the equatorial bulge has a small torque component at right angles to the ecliptic. This fact accounts for the slightly wavy nature (called ***nutation***) of the cone of precession of the earth's axis.

The gyro compass is not complicated in principle. However, in practice many intricate devices are necessary in the design of a gyroscope to harness its stability and precession characteristics so that it will act as an easily used, as well as an accurate, compass. Since the present aim is merely to understand generally how the principles of the gyroscope can be employed to make it a compass, complete descriptions of the remarkable solutions of the many engineering problems in the actual design of a gyro compass will be omitted. The following description, which illustrates the principles involved, is that of a simplified, theoretical gyro compass without the complicating refinements necessary for practical use. The need for some of these refinements will be pointed out at appropriate points with statements as to how they can be supplied. The student can find complete descriptions of the details of a gyro compass in books devoted to this instrument.

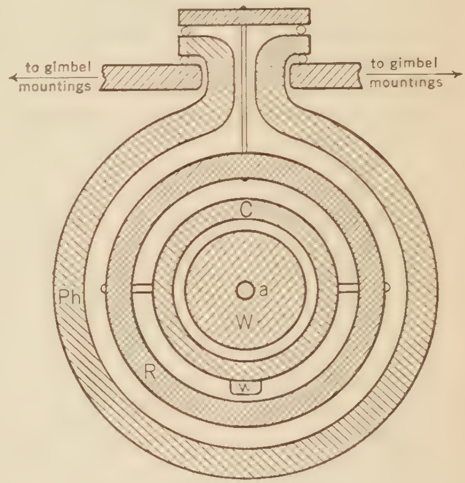


FIGURE 225

The gyro compass (see Figure 225) consists essentially of a heavy wheel, W, around the armature of an electric motor. This wheel is mounted in a casing, C, and made to spin with its axis, a, horizontal at several thousand revolutions a minute. The casing is pivoted to a vertical ring, R, along that horizontal through the center of gravity of the wheel and casing which is perpendicular to the axis of the wheel. The vertical ring is suspended from supports by a torsionless wire which allows the ring and wheel casing to rotate about the vertical axis. Attached to the bottom of the wheel casing is a small weight, w, which, by the attraction of gravity, tends to keep the axis of the wheel horizontal.* The rotation of the wheel casing and vertical supporting ring about the vertical wire is sensitively duplicated through electrical contacts by a frame, called the phantom and labeled Ph, placed around the vertical ring on which the wheel casing is pivoted. The compass card is placed on this phantom and therefore turns in exact accordance

* See the refinement of w mentioned in the description of Figure 227.

with the turning of the axis of the gyro about the vertical suspending wire.

About three hours before a ship gets under way, the axes of the gyro compasses are set by hand to point approximately true north by reference to a magnetic compass and the motors are set in motion from west to east to agree with the sense of the rotation of the earth. Suppose, as is indicated by (1) in Figure 226, that the gyro compass is in northern latitude and that the axis is set pointing to a fixed star on the horizon a little to the east of north. This star will then *appear* to be rising on its westward path about the pole. Because of the stability characteristic of the gyro, its axis will tend to maintain its direction in space. As the earth rotates, the axis will tend to tilt upward out of the horizontal to (2) in order to follow this star.* As soon as the weight w at the bottom of the gyro casing is slightly elevated, the earth's attraction on this elevated weight will provide a torque tending to rotate the north end of the gyro's axis downward. The axis of this torque — according to the right-hand screw notation — would point westward. Since the gyro's axis points northward, the precessional characteristic of the gyro will make the axis precess westward. When the axis has thus passed west of north the situation described above is reversed: Imagine the gyro's axis to be pointing directly at a star close to the horizon a little to the west of north. This star will then appear to be sinking to the east. As the gyro seeks to follow this star its axis will have to tilt with the north end down, thereby bringing into play the torque of gravity, which now has its axis to the east. Hence, the gyro's axis will precess to the east. The axis of the gyro then successively coincides with the elements of an elliptical cone with vertex at the center of the gyro and axis of the cone horizontal in the direction of true north.

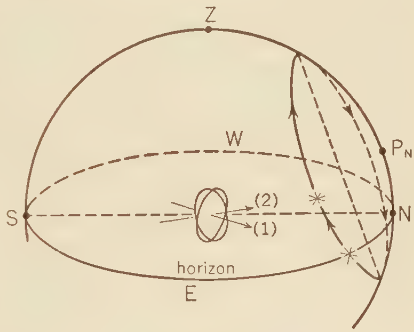


FIGURE 226

About eighty-four minutes are required for the gyro to complete

* This is, of course, an illusion due to the earth's axial rotation (see chapter 5). The star is fixed in space; and, because of the earth's axial rotation, the observer's horizon tilts with respect to the line of sight of this star. Hence, instead of the axis of the gyro tilting up out of the horizontal, the horizontal is actually tilted *down* away from the axis of the gyro. In any case, the ensuing gravity torque tends to pull the north end of the axis of the gyro compass back into the horizontal.

a cycle of these oscillations about the geographic north. Hence, unless the amplitude of these oscillations is damped, the immediate usefulness of the gyro compass is limited. This damping is successfully accomplished by various refinements. One of these involves removing the weight w from fixed attachment to the wheel casing and pivoting it to the phantom at points above the weight's center of gravity so that it tends to hang vertically down underneath the wheel casing as is suggested in Figure 227.* This suspended weight is then attached to the wheel casing by a pin, p. Hence, as soon as the gyro's axis tilts upward in either direction, the weight bears against the pin on the wheel casing and thereby provides the precessing torque. By having the pin p mounted slightly eccentrically to the wheel casing the weight w provides, in addition to a precessing torque, a damping torque in the horizontal plane.

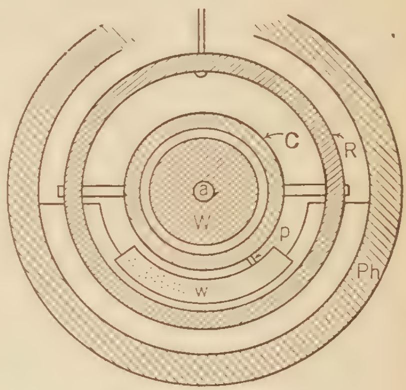


FIGURE 227

The phantom casing containing the gyro wheel is mounted in gimbels in the binnacle fixed to the ship, much as a magnetic compass is mounted. Because of the complexity and sensitivity of the gyro compass, the equipment described above is firmly fixed to the ship in a protected room below decks. ***Repeater compasses*** are then set up on the bridge, in the engine room, and anywhere else desired. These are merely compass cards which, by means of electrical relay devices, rotate exactly as does the compass card of the master gyro compass.

31. The Directional Gyro in Aerial Navigation

The stability characteristic of a gyroscope is used in a much simpler type of gyro compass than the above. This is the directional gyro used in aircraft where the speeds and banked turns make the magnetic compass and the gyro compass impractical.

The directional gyro consists of a wheel kept rotating at high speed by air currents. Being set in gimbels allowing complete freedom of motion, its axis maintains its original direction. This direction is

* Slots are provided in the vertical supporting ring through which the pivots connecting the weight w to the phantom can run.

compared with a magnetic compass when the airplane is traveling at a constant speed on a straight course, and the scale on the directional gyro is accordingly set by hand so that thereafter the directional gyro will indicate the compass heading of the airplane. The directional gyro is unaffected by turns or changes of speed. It is checked with the magnetic compass every half hour or oftener. The scale on the directional gyro is on a vertical cylinder, part of which is visible through a window. The compass heading is the scale reading at the pointer in the center of the window. Since the directional gyro does not have a horizontal compass card, this compass cannot be used with the azimuth circle.

32. The Pelorus

The pelorus is merely a dummy compass. It is a compass card mounted in gimbels and can be set by hand as desired. By setting the pelorus compass card so that its lubber's line indicates the same heading as the magnetic compass card or a gyro repeater card, bearings and azimuths can be taken with the pelorus while the ship's head is maintained. A pelorus is obviously a very simple instrument. When set up in an exposed position commanding clear views of the horizon it is frequently more useful for observing bearings and azimuths than an actual compass card used for steering on the bridge.

PART FOUR: *The Transit and Its Solar Attachment*

33. General Description of the Engineer's Transit

Though generally used to obtain data for problems in plane trigonometry, the engineer's transit is used also for observing spherical trigonometry data. Its use is confined to observations on land, as the instrument must be kept perfectly level.

Figure 228 illustrates the essential features of a transit. Basically a transit is a telescope, T, mounted on a tripod so that it can be turned in a vertical plane about a horizontal axis, $H.A.$, perpendicular to the telescope's axis and also in the horizontal plane about a vertical axis, $V.A.$, intersecting the telescope's axis. These two axes of rotation are concurrent with the axis of the telescope at a point which is directly over the point — as indicated by a plumb bob — on the earth for which the observations are to be made. The amount of rotation in each plane — the horizontal and the vertical — is measured on a circular scale — $H.S.$ and $V.S.$, respectively — which is provided with a vernier, V. The horizontal circular scale can be set with its zero pointing in any desired direction, and the vertical scale can be moved sufficiently to keep the zero at the pointer when the telescope is horizontal. Level tubes, $L.T.$, are mounted on the horizontal table on which the telescope is mounted to ensure that horizontal angles are measured in the horizontal plane. A level tube, $L.T.$, is mounted also along the axis of the telescope to indicate the horizontal from which the vertical angles can be measured. A magnetic compass is usually provided and is placed on the horizontal platform on which the telescope is mounted. Numerous adjustments must be checked to ensure the accuracy of observations. Although these details are of the utmost importance when actually making observations, a general description of how the transit can be used for astronomical observations will suffice here.

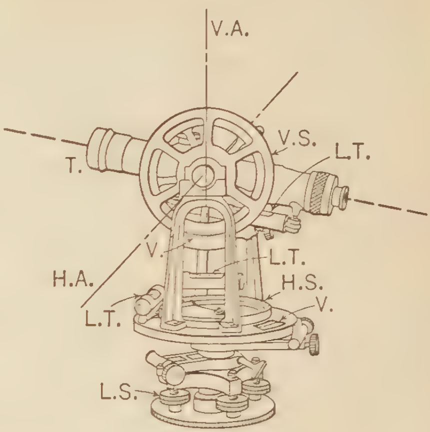


FIGURE 228

34. Altitude Observations with the Transit

The platform on which the telescope is mounted is first leveled by turning leveling screws, $L.S.$ The telescope is then turned on its vertical axis and inclined on its horizontal axis until the particular heavenly body whose altitude is desired is sighted. Further horizontal motion of the telescope can then be prevented by a screw. The angle read on the vertical circle, through which the telescope has been elevated from its level position, is the altitude of the observed heavenly body. When the sun is being observed, the observer's eye must be protected from the focused rays of the sun. A dark glass or a prism reflecting the image of the sun onto a ground glass face can be used. Or the image of the sun and the telescope cross hairs can be focused on a card held behind the eyepiece. Altitude observations with the transit can generally be read to minutes. A theodolite, which is merely a larger and more accurate transit, can be read to ten seconds.

35. Azimuth Observations with the Transit

If the above procedure for altitude observations is preceded by setting the zero of the horizontal scale in the direction of true north, then the reading on this horizontal scale when the celestial body is sighted will be the azimuth of the body. This initial setting of the horizontal scale can be accomplished by clamping the platform of the telescope to the circular scale when the axis of the telescope is in line with the zero of this scale — as shown by the vernier — and then rotating the telescope with the horizontal scale fixed to its platform until the telescope is sighted in the direction of true north. This latter can be determined by sighting on Polaris or on any other star at the instant it is known to be on the meridian, by sighting on some station previously established to lie on the meridian, or by setting the zero of the horizontal scale to true north by means of the magnetic compass corrected for variation and deviation. Azimuth observations can generally be read to thirty seconds with the transit and to ten seconds with a theodolite.

36. The Solar Attachment to a Transit

A spherical trigonometry problem which frequently arises in surveying is that of locating the meridian. The most accurate procedure is to use an engineer's transit to sight on Polaris when, by reference to tables from the *Nautical Almanac*, it is known to be at an ***elongation***, that is, at the most eastern or most western point on its small circle path about the pole. Then, using data from the *Nautical Almanac* (see Appendix IV), the observer can solve a *PZM* * spherical triangle to give the azimuth of Polaris at this instant (that is, the angle through which the transit telescope must be turned from the direction of sighting on Polaris to set a stake due north of the point over which the transit has been placed).† This procedure involves observations at night and the illumination of the cross hairs of the telescope. For daytime observation a method nearly as accurate as the one above is that of solving a *PZM* spherical triangle arising from a transit observation of the sun at a known instant of Greenwich time and from data in the *Nautical Almanac*. Though it involves some loss of accuracy the solution of the *PZM* triangle arising from a solar observation can be obtained *automatically* by means of an ingenious device called the ***solar attachment to the transit***. Although solar attachments are now infrequently used (because of the greater accuracy to be obtained by logarithmic or hand book solutions of the *PZM* triangle arising from stellar or solar observations with the engineer's transit), the principles of the solar attachment are described below for the light that this instrument throws on the solid geometry aspects of the *PZM* astronomical triangle.

* The standard astronomical triangle whose vertices are a pole, the zenith, and the observed celestial body, respectively.

† This method of solution of the *PZM* triangle can be adequately approximated by tabulated data from the *Nautical Almanac* provided in a pocket-sized booklet *Solar Ephemeris* published for Keuffel and Esser of Hoboken, New Jersey.

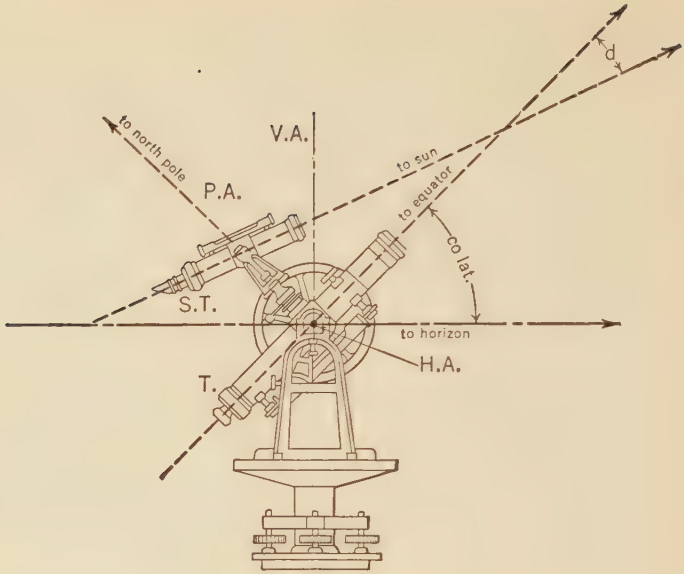


FIGURE 229

A side elevation of a standard type of solar attachment to the transit is sketched in Figure 229. It consists of an additional telescope mounted on top of the transit telescope, *T*. The additional telescope is called the solar telescope, *S.T.* Like the transit telescope, the solar telescope can be turned in azimuth and altitude. But, whereas the axis, *V.A.*, about which the transit telescope rotates in azimuth is always vertical, the corresponding axis about which the solar telescope rotates, the *polar axis*, *P.A.*, is always perpendicular both to the axis of the transit telescope, that is, the line of sight of the telescope, and also to the horizontal axis, *H.A.*, about which the transit telescope rotates in altitude. Consequently, if a point of observation in northern latitude (compare Figure 230) is assumed, this polar axis of the solar telescope will point to the north celestial pole (from which this axis takes its name) when the transit telescope is elevated from the south point on the horizon by an amount equal to the co-latitude of the point of observation; and vice versa. The transit telescope will, therefore, be put into the plane of the meridian whenever the polar axis is pointed to the celestial pole. The polar axis is pointed to the celestial pole by means of the solar telescope as is described below.

In Figure 230 the sun is assumed to have southern declination. When the solar telescope is inclined to the polar axis by 90° + the southern declination of the sun, the solar telescope can then be made to follow the sun over its entire path for the day by rotating the solar telescope about the polar axis, *provided* this axis is directed toward the celestial pole. The converse of this statement is also true: When the solar telescope has been placed, by turning the polar axis about the vertical axis of the transit telescope, so that in this position of the polar axis the solar telescope can follow the sun throughout the day, then the

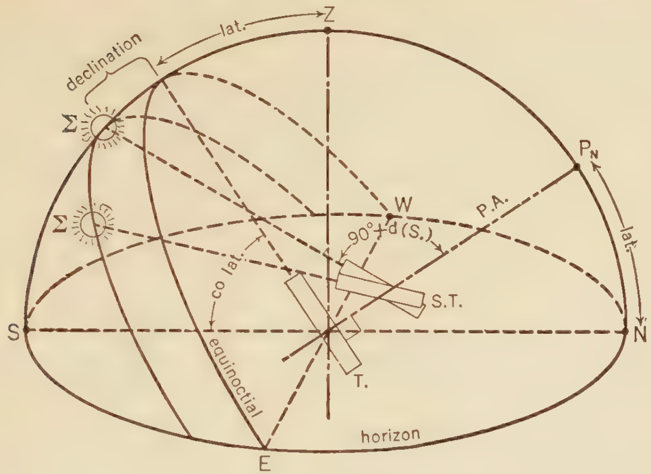


FIGURE 230

polar axis will be pointing to the celestial pole. It would obviously be inconvenient to check this setting by watching the sun through the solar telescope over the sun's entire visible path for the day. It is furthermore unnecessary to do this if the following ***rule of azimuth order of the two telescopes for northern latitudes*** is obeyed:

*For **forenoon** observations the solar telescope must point to the **left** of the transit telescope.*

*For **afternoon** observations the solar telescope must point to the **right** of the transit telescope.*

The justification of the rule is immediately seen when it is recalled that when the transit is in a position so that the meridian can be located, the transit telescope is pointing south. The sun must then be to the left of the transit telescope in the forenoon and to the right in the afternoon.

THEOREM: ***Procedure for determining the meridian in known northern latitude * by means of the solar attachment to a transit.***

1. *If the transit platform is leveled over the point of observation and the transit telescope leveled, and*

2. *If the solar telescope is inclined out of the horizontal by the sun's declination, d, (below the horizontal, if southern; above the horizontal, if northern) and this setting fixed, and*

3. *If the transit telescope is elevated from the horizontal by the co-latitude of the point of observation and this setting fixed* (see Figure 229), *and*

4. *If the two telescopes are rotated about the polar axis and the vertical axis, respectively, until the sun is visible in the solar telescope* when *this solar telescope is pointing to the* left *of the transit telescope in the* forenoon *or to the* right *in the* afternoon, *and the transit telescope then fixed in azimuth,* (see Figure 230),

Then: The transit telescope will be in the vertical plane of the meridian.

Hypothesis 2 is, of course, equivalent to the condition given above,

* The changes in this procedure for southern latitude are obvious.

namely, that the solar telescope be inclined by 90° + the sun's southern declination with the polar axis. Thus, at this stage the two telescopes are inclined to one another by the sun's declination, the solar telescope pointing down toward the transit telescope for southern declination and up away from the transit telescope for northern declination.

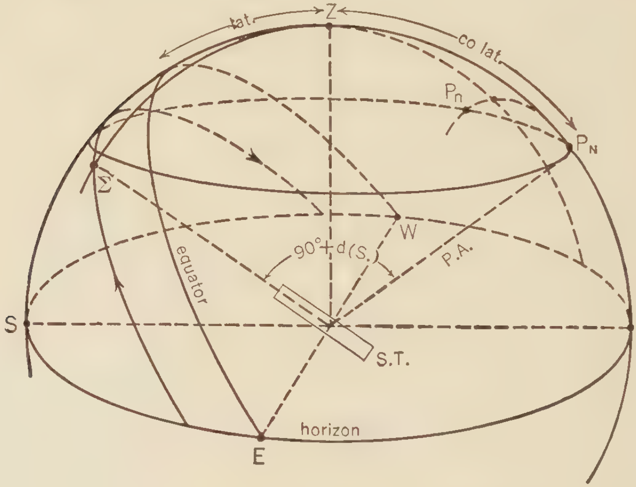
The force of hypothesis 4, which is merely the rule of azimuth order of the two telescopes stated above, is that but *one* observation of the sun is necessary.

Proof:

1. The north celestial pole must lie on each of two small circles of the celestial sphere:

   a. Because the polar axis has been free to rotate around the fixed vertical axis, to which it is inclined at the co-latitude of the point of observation, the north pole must lie on the small circle with the observer's zenith as pole and the co-latitude of the observer as polar distance.

   b. Because the polar axis is set to be inclined at 90° minus the sun's signed declination (that is, positive when north; negative when south) to the solar telescope, the north pole must lie on the small circle with the observed position of the sun as pole and 90° minus the sun's signed declination as polar distance.



Σ = sun. S.T. = solar telescope. P.A. = polar axis.

FIGURE 231

2. These two small circles will, in general, meet in two distinct points, P_N and P_n (see Figure 231), each of which will then be a point of possible direction of the polar axis, but only one of which can be the north celestial pole.

3. By Introduction, 7 *c*, P_N and P_n must lie on opposite sides of the

great circle, ΣZ, through the observed position of the sun and through the observer's zenith.

4. From Figure 231, since the transit telescope must be in the vertical plane of either P_N or P_n, for one of these points the solar telescope must be pointing to one side of the transit telescope and for the other point the solar telescope must be pointing to the opposite side. Since the rule of azimuth order is obeyed when the polar axis is pointing to the celestial pole P_N, the possibility of the polar axis's pointing to P_n is obviated when this rule is obeyed.

q.e.d.

PART FIVE: *The Radio Direction-Finder*

37. Properties of the Loop Antenna

When the antenna of a radio receiving station is in the shape of a plane loop the intensities of signals received by this antenna are observed to vary according to the angle at which the *oncoming signals meet the plane of the antenna.* Specifically, the intensity is least when the signals are received normal to the plane of the antenna, and the intensity is greatest when the signals are received parallel to the plane of the antenna. This phenomenon is of use in determining the bearings of a ship from a properly equipped radio station receiving signals from the ship. The loop antenna of the receiving station is rotated on a vertical axis while the ship's signals are being received until the position of least intensity is obtained. Then the direction of the normal to the plane of the antenna is that from which the ship's signals were received. The radio station accordingly transmits to the ship the ship's radio bearing, where by

DEFINITION: *The **ship's radio bearing** from the radio station is the true compass course of the ship's signals as received at the radio station, plus or minus* 180°.

The radio equipment of such a radio station is termed a ***radio direction-finder.*** Such stations are established at strategic coastal points. At all times, except during the first ten minutes of every hour in clear weather, the stations are available to test for fifty seconds the oncoming direction of the radio signals of any ship so signaling for this service, and then to transmit the ship's thus determined radio bearings. In practice several corrections must be applied to the readings of a radio direction-finder to give the correct radio bearing of a signaling ship. Radio bearings are accurate to within about 2° and for distances up to about 150 miles.

38. Great-Circle Paths of Radio Signals

Since radio signals from a ship to a radio direction-finder station travel on a geodesic or minimum distance path on the spherical shell immediately surrounding the earth, they follow *the great-circle path from the ship to the station.* Consequently, the radio direction-finder is an instrument by which data for spherical triangle solutions may be observed.

For distances of less than fifty miles the ship's radio bearing from the station can be used with adequate accuracy on a Mercator chart to draw the ship's line of position from the station. On a Mercator chart all straight lines are rhumb lines, that is, lines of constant true bearing. The radio bearing of a ship is *the direction* at the radio station *of the great-circle path to the ship.* However, for distances of less than fifty miles there is generally little change in the direction along a great circle.

For distances of fifty miles or more three procedures are possible in using the radio bearing of a ship from a radio station to determine the position of the ship:

1. From the known position of the station and an *estimated* position of the ship apply certain tabulated corrections to the radio bearing to give the Mercator bearing; draw this line of Mercator bearing of the ship from the station to give a line of position of the ship; repeat this procedure for the radio bearing from another station to obtain the intersection of two lines of position as the ship's position.

2. Draw the radio bearing lines from two stations on radio direction-finder charts, which are available for certain localities. These are charts on such a projection as to make lines of radio bearings straight lines.

3. ***Spherical trigonometry solution:*** Consider the ship as lying on that great circle from the station which is uniquely described by the ship's radio bearing from this station. Then by combining this datum with that of some other observation, such as a meridian altitude of the sun, or a star to determine latitude, or a radio bearing from another station, find the ship's position by the solution of one or more spherical triangles. (This procedure, though theoretically possible, is seldom used in practice. See problems 13 and 14 in section 38, Chapter 4.)

39. Radio Direction-Finders on Ships and Planes

Radio direction-finders can, of course, be carried on ships and planes. Then the radio bearing *at the ship or plane of a radio station* can be ascertained immediately, which, in the case of a plane, is particularly important, because of the speed with which a plane changes position. If a ship carrying a radio direction-finder obtains both *its own* radio bearing *from a radio station* in known latitude and longitude and also *the radio station's bearing at the ship*, then the course at both ends of the great-circle arc between the ship and station are known, and therefore sufficient data are available for the solution of the s.a.a. ambiguous spherical triangle. If the solution is unique, or if some additional qualitative information is available to dispose of one of two possible solutions, the ship's position can be obtained by the solution of this triangle.

The radio direction-finder on a plane is particularly useful for "homing," that is, for directing the plane to a ship or station sending out signals. The plane of the loop antenna is kept fixed in a position normal to the fore-and-aft axis of the airplane. The plane is then so directed that no signals are heard from the home station. In this way the airplane is guided home on the great-circle path. The plane must periodically be turned off its course to see whether the home station is still transmitting.

APPENDIX IV

The *Nautical Almanac* and the *Air Almanac*

General Description of the *Nautical Almanac* and the *Air Almanac*

Frequent references have been made in the text to the two almanacs, the *American Nautical Almanac* and the *American Air Almanac*.* These publications record the celestial coordinates of the more conspicuous heavenly bodies, both those which are fixed and those which vary according to the argument Greenwich civil time. It is by reference to these known quantities, independent of the observer, that an observation dependent on the observer can give the otherwise unknown position or time of the observation.

The *American Nautical Almanac* is much the older publication and is standard at sea. It is more accurate and more complete than the *American Air Almanac*, which was first published in 1941 and is standard for air navigation. The arrangements of these two almanacs are entirely dissimilar. Because the arrangement of the *Air Almanac* is such that a great variety of celestial problems can be touched upon in a small space, all but one of the excerpts which follow are from the *Air Almanac*. The table on Polaris is from the *Nautical Almanac*.

These excerpts from the *Air Almanac* include, either directly or by inference, the essence of this almanac. The tables involving the corrections for dip, refraction, etc., have been omitted, as these corrections are to be omitted in the problems in this text. In the explanations of the *Air Almanac*, also copied from this publication, the exercises have been altered to fit the daily sheet (that for August 1, 1943) here reproduced. These changes and an addition in the explanation of the star chart are indicated by brackets. Any omission in the explanations is indicated by a row of asterisks. A footnote calls attention to the omission of certain symbols from the table of stars.

* Both these almanacs are issued by The Nautical Almanac Office at the United States Naval Observatory in Washington, D.C. They are available from the Superintendent of Documents, Washington, D.C. The *Nautical Almanac* is published in one volume for each year at sixty-five cents a volume. The *Air Almanac* is published in three volumes (one for each third of a year) at one dollar a volume of four months. Each volume of these almanacs is available several months before it becomes current.

Permission to reproduce these almanac excerpts has been graciously granted by Captain J. F. Hellweg, U.S.N. (Retired), Superintendent of the United States Naval Observatory.

THE AMERICAN AIR ALMANAC

EXPLANATION AND EXAMPLES

* * * * * *

Columns 2–7 of the daily sheets give the Greenwich Hour Angles at ten-minute intervals for the Sun, Vernal Equinox, the three planets most suitable for observation at that time, and the Moon, and declinations at ten-minute intervals for the Moon and at hourly intervals for the Sun and planets. The magnitudes of the planets are given in the headings with their names.

* * * * * *

The GHA of a star is found by adding the Greenwich Hour Angle of the Vernal Equinox to the star's Sidereal Hour Angle; i.e.,

$$\text{GHA} * = \text{GHA} \Upsilon + \text{SHA} *$$

On the inside of the back cover [page 242 in book] are given the Name, Mag., SHA, Dec.,† and RA of each of the 55 principal navigational stars; stars brighter than magnitude 1.5 and Polaris are given in bold type [italic type is used in this book.] Two separate lists are given: one in alphabetical order, and the other in order of SHA.

* * * * * *

[EXAMPLE: For Aug. 1, 1943, at $17^h 47^m 16^s$ find the GHA and Dec. of Canopus.

(From the daily sheet for Aug. 1) GHA ♈ at $17^h 40^m$		214° 27′
(From table for Interpolation of GHA) GHA ♈ Int. $7^m 16^s$		1° 49′
(From Star data) SHA *		264° 20′
The required GHA *	120° 36′ =	480° 36′
(From Star data) The required Dec.		S52° 40′]

The semidiameters of the Sun and Moon and the correction for Moon's parallax are given on the A.M. side of the daily sheets; the values given are for 12^h GCT and the first correction value in the parallax table is the Moon's Horizontal Parallax.

* * * * * *

† Dec., or *declination*, is the equivalent of d in the rest of the text.

The diagram on the A.M. side of the daily sheet shows the region of the sky along the ecliptic circle within which the Sun, Moon, and planets always appear. The positions of the Moon, the five planets, Mercury, Venus, Mars, Jupiter, Saturn, and the four bright stars Aldebaran, Antares, Spica, and Regulus are shown, except when they are within 5° of the Sun. The position of the Vernal Equinox (♈) is also shown for reference.

The symbols used to show the positions of the various objects also indicate their appearance. The Moon symbol shows the correct phase, and the symbols for the stars and planets indicate brightness or magnitude. The stars are all of the first magnitude and are indicated by the symbol *. A planet of first magnitude is indicated by a large dot and one of second magnitude by a small dot. Planets brighter than the first magnitude are represented by a circle with a number of small marks attached, the brighter the object the more marks; magnitudes 0, −1, −2, −3, −4 being represented by a circle with 0, 1, 2, 3, 4 marks, respectively.

The Sun is always shown in the center of the diagram and the attached scale shows angular distance from the Sun. The diagram is 360° long and actually represents a complete circle around the sky, the two ends of the diagram representing the point on the sky 180° from the Sun.

At any given time only about half the region on the diagram is above the horizon. At Sunrise, the Sun (and hence the region near the center of the diagram) is rising in the East, and the region at the end marked "West" is setting in the West. The region halfway between is on the meridian. At the time of Sunset, on the other hand, the Sun is setting in the West and the region at the end marked "East" is rising in the East.

[Example: At sunrise on August 1, 1943, one finds from the diagram that the Moon is so far gone in the last quarter as to be invisible. Mars, a zero magnitude object, is just east of the meridian. Aldebaran is 30° east of the meridian, and Saturn, of zero magnitude, is halfway between the meridian and eastern horizon and slightly north of Aldebaran. At sunset, on the other hand, one finds Mercury (of magnitude −1) and Regulus close to the Sun, and Venus (of magnitude −4) about 40° east of the sun. Spica is about 15° west and Antares about 30° east of the meridian. The latter will be favorable for observation most of the night. Shortly after midnight Mars will rise in the east and be visible until dawn.]

* * * * * *

Tables for finding the times of Sunrise, Sunset, beginning and ending of Civil Twilight, Moonrise, and Moonset for latitudes between 60° S and

70° N are given on the P.M. side of the daily sheets. The columns under Sunrise and Sunset give the local civil times of these phenomena. The columns under Twilight (Twlt.) give the duration of Civil Twilight. It is assumed that morning Civil Twilight begins when the Sun is 6° below the horizon and ends at Sunrise and that evening Civil Twilight begins at Sunset and ends when the Sun is 6° below the horizon. The time of beginning of morning Civil Twilight is obtained by subtracting the duration of Twilight from the time of Sunrise; the ending of evening Twilight is obtained by adding the duration of Twilight to the time of Sunset.

Some of these phenomena do not occur in high latitudes during certain periods. The symbols used to indicate the exceptions are:

▭, Sun or Moon does not set but remains continuously above the horizon.

▬, Sun or Moon does not rise but remains continuously below the horizon.

////, Twilight lasts all night.

When the Sun is continuously below the horizon it may produce Twilight for a part of the day; the value then given in the Twilight column is the interval from beginning of morning Twilight to meridian passage or from meridian passage to ending of evening Twilight, the total duration of Twilight being twice the tabulated value.

*   *   *   *   *   *

The columns under Moonrise and Moonset give the Local Civil Time of these phenomena for the meridian of Greenwich. Since the times of Moonrise and Moonset are usually considerably later on succeeding days, it is necessary to interpolate for the longitude of the observer; the last column (Diff.) is provided for this purpose. The interpolation will, however, not be made when the difference is negative or when it does not exist; the symbol * is then given in the Diff. column.

[EXAMPLE: Find the time of moonrise in longitude 118° W. and latitude 32° N. on Aug. 1, 1943. By interpolation it is found that the time of moonrise for the meridian of Greenwich and latitude 32° N. is $5^h\ 22^m$. From the Diff. column it is found that the time of rising will be 53^m later the following day. Since 118° is about one-third of 360°, one adds 18^m.

$$5^h\,22^m + 18^m = 5^h\,40^m \text{ LCT.}]$$

*   *   *   *   *   *

The LCT found in the [above] is the local time for the observer's local meridian. GCT is obtained from the LCT by applying the observer's longitude from Greenwich; the longitude is first converted from arc to time and then added to the LCT for an observer in west longitude or subtracted for an observer in east longitude.

[Example: Change $5^h\ 40^m$ LCT in longitude 118° W. to GCT.

Longitude 118° W. converted to time is longitude $7^h\ 52^m$ W.;
$5^h\ 40^m$ LCT + $7^h\ 52^m = 13^h\ 32^m$ GCT.]

The tables, Interpolation of GHA, Dip, Polaris, ☾'s Par. and Corr. HA ☾, are the so-called "critical" or "turning point" type; i.e., the values of the argument given are those for which the function changes from one unit to the next. The value of the function is therefore found to the nearest unit without interpolation. If the required value of the argument is one of the printed values of the table, the upper of the two adjacent values of the function should be taken.

The error of an interpolated GHA is never as great as 1′.8, and the average error is about 0′.5, except for those circumpolar stars whose SHA's are enclosed in parentheses. †

† See footnote, page 130, and also Appendix II, page 221.

GCT	☉ SUN GHA	☉ SUN Dec.	♈ GHA	VENUS −4.2 GHA	VENUS −4.2 Dec.	MARS 0.2 GHA	MARS 0.2 Dec.	SATURN 0.3 GHA	SATURN 0.3 Dec.	● MOON GHA	● MOON Dec.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
0 00	178 26	N18 18	308 44	141 56	N 2 40	264 35	N14 58	226 42	N21 55	180 33	N17 53
10	180 56		311 14	144 26		267 05		229 13		182 59	53
20	183 26		313 45	146 57		269 35		231 43		185 24	52
30	185 56	·	316 15	149 27	·	272 05	·	234 13	·	187 49	51
40	188 26		318 45	151 57		274 36		236 44		190 15	50
50	190 56		321 16	154 27		277 06		239 14		192 40	49
1 00	193 26	N18 17	323 46	156 58	N 2 39	279 36	N14 59	241 44	N21 55	195 05	N17 49
10	195 56		326 17	159 28		282 06		244 15		197 30	48
20	198 26		328 47	161 58		284 36		246 45		199 56	47
30	200 56	·	331 17	164 28	·	287 06	·	249 16	·	202 21	46
40	203 26		333 48	166 59		289 37		251 46		204 46	45
50	205 56		336 18	169 29		292 07		254 16		207 12	45
2 00	208 26	N18 17	338 49	171 59	N 2 38	294 37	N14 59	256 47	N21 55	209 37	N17 44
10	210 56		341 19	174 29		297 07		259 17		212 02	43
20	213 26		343 50	177 00		299 37		261 47		214 28	42
30	215 56	·	346 20	179 30	·	302 07	·	264 18	·	216 53	41
40	218 26		348 50	182 00		304 37		266 48		219 18	41
50	220 56		351 21	184 30		307 08		269 18		221 44	40
3 00	223 26	N18 16	353 51	187 01	N 2 37	309 38	N15 00	271 49	N21 55	224 09	N17 39
10	225 56		356 22	189 31		312 08		274 19		226 34	38
20	228 26		358 52	192 01		314 38		276 50		228 59	37
30	230 56	·	1 22	194 31	·	317 08	·	279 20	·	231 25	37
40	233 26		3 53	197 02		319 38		281 50		233 50	36
50	235 56		6 23	199 32		322 08		284 21		236 15	35
4 00	238 26	N18 16	8 54	202 02	N 2 36	324 39	N15 00	286 51	N21 55	238 41	N17 34
10	240 56		11 24	204 32		327 09		289 21		241 06	33
20	243 26		13 54	207 03		329 39		291 52		243 31	32
30	245 56	·	16 25	209 33	·	332 09	·	294 22	·	245 57	32
40	248 26		18 55	212 03		334 39		296 52		248 22	31
50	250 56		21 26	214 34		337 09		299 23		250 47	30
5 00	253 26	N18 15	23 56	217 04	N 2 36	339 39	N15 01	301 53	N21 55	253 13	N17 29
10	255 56		26 27	219 34		342 10		304 24		255 38	28
20	258 26		28 57	222 04		344 40		306 54		258 03	27
30	260 56	·	31 27	224 35	·	347 10	·	309 24	·	260 29	27
40	263 26		33 58	227 05		349 40		311 55		262 54	26
50	265 56		36 28	229 35		352 10		314 25		265 19	25
6 00	268 26	N18 14	38 59	232 05	N 2 35	354 40	N15 01	316 55	N21 55	267 45	N17 24
10	270 56		41 29	234 36		357 11		319 26		270 10	23
20	273 26		43 59	237 06		359 41		321 56		272 35	22
30	275 56	·	46 30	239 36	·	2 11	·	324 26	·	275 00	21
40	278 26		49 00	242 06		4 41		326 57		277 26	21
50	280 56		51 31	244 37		7 11		329 27		279 51	20
7 00	283 26	N18 14	54 01	247 07	N 2 34	9 41	N15 01	331 58	N21 55	282 16	N17 19
10	285 56		56 31	249 37		12 11		334 28		284 42	18
20	288 26		59 02	252 07		14 42		336 58		287 07	17
30	290 56	·	61 32	254 38	·	17 12	·	339 29	·	289 32	16
40	293 26		64 03	257 08		19 42		341 59		291 58	15
50	295 56		66 33	259 38		22 12		344 29		294 23	14
8 00	298 26	N18 13	69 03	262 08	N 2 33	24 42	N15 02	347 00	N21 55	296 48	N17 14
10	300 56		71 34	264 39		27 12		349 30		299 14	13
20	303 26		74 04	267 09		29 42		352 00		301 39	12
30	305 56	·	76 35	269 39	·	32 13	·	354 31	·	304 04	11
40	308 26		79 05	272 10		34 43		357 01		306 30	10
50	310 56		81 36	274 40		37 13		359 32		308 55	09
9 00	313 26	N18 13	84 06	277 10	N 2 32	39 43	N15 02	2 02	N21 55	311 20	N17 08
10	315 56		86 36	279 40		42 13		4 32		313 46	07
20	318 26		89 07	282 11		44 43		7 03		316 11	07
30	320 56	·	91 37	284 41	·	47 13	·	9 33	·	318 36	06
40	323 26		94 08	287 11		49 44		12 03		321 02	05
50	325 56		96 38	289 41		52 14		14 34		323 27	04
10 00	328 26	N18 12	99 08	292 12	N 2 31	54 44	N15 03	17 04	N21 55	325 53	N17 03
10	330 56		101 39	294 42		57 14		19 34		328 18	02
20	333 26		104 09	297 12		59 44		22 05		330 43	01
30	335 56	·	106 40	299 42	·	62 14	·	24 35	·	333 09	· 17 00
40	338 26		109 10	302 13		64 45		27 06		335 34	16 59
50	340 56		111 40	304 43		67 15		29 36		337 59	58
11 00	343 26	N18 11	114 11	307 13	N 2 30	69 45	N15 03	32 06	N21 55	340 25	N16 58
10	345 56		116 41	309 43		72 15		34 37		342 50	57
20	348 26		119 12	312 14		74 45		37 07		345 15	56
30	350 56	·	121 42	314 44	·	77 15	·	39 37	·	347 41	55
40	353 26		124 13	317 14		79 45		42 08		350 06	54
50	355 56		126 43	319 45		82 16		44 38		352 31	53
12 00	358 26	N18 11	129 13	322 15	N 2 30	84 46	N15 04	47 08	N21 55	354 57	N16 52

☾'s Par.

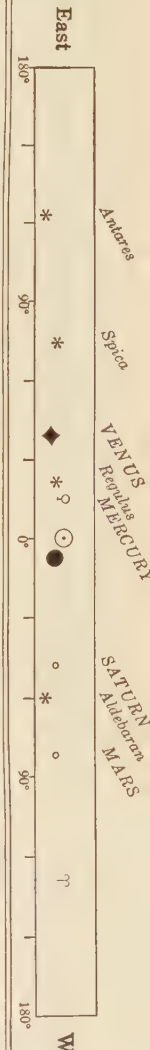
Alt. °	Corr. + ′
0	54
7	53
13	52
17	51
20	50
23	49
26	48
28	47
30	46
32	45
34	44
36	43
38	42
39	41
41	40
42	39
44	38
45	37
47	36
48	35
50	34
51	33
52	32
54	31
55	30
56	29
58	28
59	27
60	26
61	25
63	24
64	23
65	22
66	21
67	20
68	19
69	18
71	17
72	16
73	15
74	14
75	13
76	12
77	11
78	10
79	

SD ☉ ′ 16

SD ☾ ′ 15

Corr. HA ☾

Int. m	Corr. ′
0	0
10	



GREENWICH P.M. 1943 AUGUST 1 (SUNDAY)

GCT	☉ SUN GHA	SUN Dec.	♈ GHA	VENUS −4.2 GHA	VENUS Dec.	MARS 0.2 GHA	MARS Dec.	SATURN 0.3 GHA	SATURN Dec.	● MOON GHA	MOON Dec.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
12 00	358 26	N18 11	129 13	322 15	N 2 30	84 46	N15 04	47 08	N21 55	354 57	N16 52
10	0 56		131 44	324 45		87 16		49 39		357 22	51
20	3 26		134 14	327 15		89 46		52 09		359 47	50
30	5 56		136 45	329 46		92 16		54 40		2 13	49
40	8 26		139 15	332 16		94 46		57 10		4 38	48
50	10 56		141 45	334 46		97 16		59 40		7 03	47
13 00	13 26	N18 10	144 16	337 16	N 2 29	99 47	N15 04	62 11	N21 55	9 29	N16 46
10	15 56		146 46	339 47		102 17		64 41		11 54	45
20	18 26		149 17	342 17		104 47		67 11		14 19	45
30	20 56		151 47	344 47		107 17		69 42		16 45	44
40	23 26		154 17	347 17		109 47		72 12		19 10	43
50	25 56		156 48	349 48		112 17		74 42		21 36	42
14 00	28 26	N18 09	159 18	352 18	N 2 28	114 48	N15 05	77 13	N21 55	24 01	N16 41
10	30 56		161 49	354 48		117 18		79 43		26 26	40
20	33 26		164 19	357 18		119 48		82 14		28 52	39
30	35 56		166 50	359 49		122 18		84 44		31 17	38
40	38 26		169 20	2 19		124 48		87 14		33 42	37
50	40 56		171 50	4 49		127 18		89 45		36 08	36
15 00	43 26	N18 09	174 21	7 20	N 2 27	129 48	N15 05	92 15	N21 55	38 33	N16 35
10	45 56		176 51	9 50		132 19		94 45		40 58	34
20	48 26		179 22	12 20		134 49		97 16		43 24	33
30	50 56		181 52	14 50		137 19		99 46		45 49	32
40	53 26		184 22	17 21		139 49		102 16		48 15	31
50	55 56		186 53	19 51		142 19		104 47		50 40	30
16 00	58 26	N18 08	189 23	22 21	N 2 26	144 49	N15 06	107 17	N21 55	53 05	N16 29
10	60 56		191 54	24 51		147 19		109 48		55 31	28
20	63 26		194 24	27 22		149 50		112 18		57 56	27
30	65 56		196 54	29 52		152 20		114 48		60 21	26
40	68 26		199 25	32 22		154 50		117 19		62 47	25
50	70 56		201 55	34 52		157 20		119 49		65 12	24
17 00	73 26	N18 08	204 26	37 23	N 2 25	159 50	N15 06	122 19	N21 55	67 38	N16 23
10	75 56		206 56	39 53		162 20		124 50		70 03	22
20	78 26		209 26	42 23		164 50		127 20		72 28	21
30	80 56		211 57	44 54		167 21		129 50		74 54	20
40	83 26		214 27	47 24		169 51		132 21		77 19	19
50	85 56		216 58	49 54		172 21		134 51		79 44	18
18 00	88 26	N18 07	219 28	52 24	N 2 24	174 51	N15 07	137 22	N21 55	82 10	N16 17
10	90 56		221 59	54 55		177 21		139 52		84 35	16
20	93 26		224 29	57 25		179 51		142 22		87 01	15
30	95 56		226 59	59 55		182 22		144 53		89 26	14
40	98 26		229 30	62 25		184 52		147 23		91 51	13
50	100 56		232 00	64 56		187 22		149 53		94 17	12
19 00	103 26	N18 06	234 31	67 26	N 2 24	189 52	N15 07	152 24	N21 55	96 42	N16 11
10	105 56		237 01	69 56		192 22		154 54		99 07	10
20	108 27		239 31	72 26		194 52		157 24		101 33	09
30	110 57		242 02	74 57		197 22		159 55		103 58	08
40	113 27		244 32	77 27		199 53		162 25		106 24	07
50	115 57		247 03	79 57		202 23		164 56		108 49	06
20 00	118 27	N18 06	249 33	82 28	N 2 23	204 53	N15 08	167 26	N21 55	111 14	N16 05
10	120 57		252 03	84 58		207 23		169 56		113 40	04
20	123 27		254 34	87 28		209 53		172 27		116 05	03
30	125 57		257 04	89 58		212 23		174 57		118 31	02
40	128 27		259 35	92 29		214 53		177 27		120 56	01
50	130 57		262 05	94 59		217 24		179 58		123 21	16 00
21 00	133 27	N18 05	264 36	97 29	N 2 22	219 54	N15 08	182 28	N21 55	125 47	N15 59
10	135 57		267 06	99 59		222 24		184 58		128 12	58
20	138 27		269 36	102 30		224 54		187 29		130 38	57
30	140 57		272 07	105 00		227 24		189 59		133 03	56
40	143 27		274 37	107 30		229 54		192 30		135 28	55
50	145 57		277 08	110 00		232 24		195 00		137 54	54
22 00	148 27	N18 04	279 38	112 31	N 2 21	234 55	N15 09	197 30	N21 55	140 19	N15 53
10	150 57		282 08	115 01		237 25		200 01		142 45	52
20	153 27		284 39	117 31		239 55		202 31		145 10	51
30	155 57		287 09	120 02		242 25		205 01		147 35	49
40	158 27		289 40	122 32		244 55		207 32		150 01	48
50	160 57		292 10	125 02		247 25		210 02		152 26	47
23 00	163 27	N18 04	294 40	127 32	N 2 20	249 56	N15 09	212 32	N21 55	154 52	N15 46
10	165 57		297 11	130 03		252 26		215 03		157 17	45
20	168 27		299 41	132 33		254 56		217 33		159 42	44
30	170 57		302 12	135 03		257 26		220 04		162 08	43
40	173 27		304 42	137 33		259 56		222 34		164 33	42
50	175 57		307 13	140 04		262 26		225 04		166 59	41
24 00	178 27	N18 03	309 43	142 34	N 2 19	264 56	N15 10	227 35	N21 55	169 24	N15 40

Lat.	Sun-rise	Twlt.	Moon-rise	Diff.
N °	h m	m	h m	m
70	1 17	////	1 57	99
68	2 10	////	2 35	87
66	2 41	////	3 02	79
64	3 05	92	22	75
62	23	73	39	70
60	38	59	3 53	68
58	3 51	53	4 04	66
56	4 01	47	14	65
54	11	43	24	62
52	20	41	31	62
50	27	37	39	60
45	44	33	4 54	57
40	4 57	30	5 07	55
35	5 08	27	17	54
30	18	25	26	53
20	35	24	43	49
10	5 49	23	5 56	48
0	6 03	22	6 10	46
10	16	22	23	43
20	30	23	37	41
30	46	25	6 53	39
35	6 55	28	7 02	38
40	7 05	30	12	36
45	18	32	25	34
50	33	36	40	31
52	39	38	47	30
54	47	41	7 54	29
56	7 56	43	8 03	27
58	8 05	46	13	25
60	8 17	50	8 24	23
S				

Lat.	Sun-set	Twlt.	Moon-set	Diff.
N °	h m	m	h m	m
70	22 47	////	22 03	*
68	21 58	////	21 37	04
66	28	////	16	11
64	21 05	89	21 00	15
62	20 47	71	20 46	18
60	33	60	34	22
58	21	54	25	23
56	10	48	16	25
54	20 00	44	08	27
52	19 52	41	20 01	28
50	44	38	19 55	29
45	28	34	41	32
40	15	30	30	34
35	19 04	27	20	36
30	18 54	25	19 12	38
20	37	24	18 57	41
10	23	23	44	43
0	18 10	22	32	45
10	17 57	22	20	47
20	43	23	18 07	50
30	27	25	17 52	52
35	18	27	43	54
40	17 07	29	33	56
45	16 55	32	21	59
50	41	35	07	61
52	34	37	17 01	62
54	26	39	16 53	64
56	17	41	45	65
58	16 08	44	35	68
60	15 57	49	16 24	70
S				

INTERPOLATION OF GHA

SUN, PLANETS, ♈

Int. (m s)	Corr. (° ′)	Int. (m s)	Corr. (° ′)	Int. (m s)	Corr. (° ′)
00 00	0 00	03 17	0 50	06 37	1 40
01	0 01	21	0 51	41	1 41
05	0 02	25	0 52	45	1 42
09	0 03	29	0 53	49	1 43
13	0 04	33	0 54	53	1 44
17	0 05	37	0 55	57	1 45
21	0 06	41	0 56	07 01	1 46
25	0 07	45	0 57	05	1 47
29	0 08	49	0 58	09	1 48
33	0 09	53	0 59	13	1 49
37	0 10	57	1 00	17	1 50
41	0 11	04 01	1 01	21	1 51
45	0 12	05	1 02	25	1 52
49	0 13	09	1 03	29	1 53
53	0 14	13	1 04	33	1 54
57	0 15	17	1 05	37	1 55
01 01	0 16	21	1 06	41	1 56
05	0 17	25	1 07	45	1 57
09	0 18	29	1 08	49	1 58
13	0 19	33	1 09	53	1 59
17	0 20	37	1 10	57	2 00
21	0 21	41	1 11	08 01	2 01
25	0 22	45	1 12	05	2 02
29	0 23	49	1 13	09	2 03
33	0 24	53	1 14	13	2 04
37	0 25	57	1 15	17	2 05
41	0 26	05 01	1 16	21	2 06
45	0 27	05	1 17	25	2 07
49	0 28	09	1 18	29	2 08
53	0 29	13	1 19	33	2 09
57	0 30	17	1 20	37	2 10
02 01	0 31	21	1 21	41	2 11
05	0 32	25	1 22	45	2 12
09	0 33	29	1 23	49	2 13
13	0 34	33	1 24	53	2 14
17	0 35	37	1 25	57	2 15
21	0 36	41	1 26	09 01	2 16
25	0 37	45	1 27	05	2 17
29	0 38	49	1 28	09	2 18
33	0 39	53	1 29	13	2 19
37	0 40	57	1 30	17	2 20
41	0 41	06 01	1 31	21	2 21
45	0 42	05	1 32	25	2 22
49	0 43	09	1 33	29	2 23
53	0 44	13	1 34	33	2 24
57	0 45	17	1 35	37	2 25
03 01	0 46	21	1 36	41	2 26
05	0 47	25	1 37	45	2 27
09	0 48	29	1 38	49	2 28
13	0 49	33	1 39	53	2 29
17	0 50	37	1 40	57	2 30
21		41		10 00	

Correction to be added to GHA for interval of GCT

STARS

Alphabetical order				Order of SHA			
Name	Mag.	SHA	Dec.	SHA	Dec.	RA	Name
		° ′	° ′	° ′	° ′	h m	
Acamar	3.4	315 59	S40 32	14 31	N14 54	23 02	Markab
Achernar	0.6	336 06	S57 31	16 22	S29 55	22 55	Fomalhaut
Acrux	1.1	174 08	S62 47	28 50	S47 14	22 05	Al Na'ir
Adhara	1.6	255 54	S28 54	34 39	N 9 37	21 41	Enif
Aldebaran	1.1	291 50	N16 24	50 07	N45 05	20 40	Deneb
Alioth	1.7	167 07	N56 16	54 42	S56 55	20 21	Peacock
Al Na'ir	2.2	28 50	S47 14	63 00	N 8 43	19 48	Altair
Alnilam	1.8	276 40	S 1 14	77 04	S26 22	18 52	Nunki
Alphard	2.2	218 48	S 8 25	81 14	N38 44	18 35	Vega
Alphecca	2.3	126 56	N26 54	84 54	S34 25	18 20	Kaus Aust.
Alpheratz	2.2	358 38	N28 47	91 10	N51 30	17 55	Etamin
Al Suhail	2.2	223 32	S43 12	96 55	N12 36	17 32	Rasalague
Altair	0.9	63 00	N 8 43	97 33	S37 04	17 30	Shaula
Antares	1.2	113 31	S26 18	103 13	S15 39	17 07	Sabik
Arcturus	0.2	146 44	N19 29	(109 20)	S68 56	16 43	α Tri. Aust.
ε Argus	1.7	234 40	S59 20	113 31	S26 18	16 26	Antares
Bellatrix	1.7	279 29	N 6 18	120 45	S22 28	15 57	Dschubba
Betelgeux	0.1–1.2	271 59	N 7 24	126 56	N26 54	15 32	Alphecca
Canopus	− 0.9	264 20	S52 40	(137 17)	N74 24	14 51	Kochab
Capella	0.2	281 53	N45 56	141 04	S60 36	14 36	Rigil Kent.
Caph	2.4	358 28	N58 50	146 44	N19 29	14 13	Arcturus
θ Centauri	2.3	149 10	S36 06	149 10	S36 06	14 03	θ Centauri
β Crucis	1.5	168 54	S59 23	159 27	S10 52	13 22	Spica
γ Crucis	1.6	173 00	S56 48	159 35	N55 14	13 22	Mizar
Deneb	1.3	50 07	N45 05	167 07	N56 16	12 52	Alioth
Deneb Kait.	2.2	349 49	S18 18	168 54	S59 23	12 44	β Crucis
Denebola	2.2	183 28	N14 53	173 00	S56 48	12 28	γ Crucis
Dschubba	2.5	120 45	S22 28	174 08	S62 47	12 23	Acrux
Dubhe	2.0	194 57	N62 04	183 28	N14 53	11 46	Denebola
Enif	2.5	34 39	N 9 37	194 57	N62 04	11 00	Dubhe
Etamin	2.4	91 10	N51 30	208 40	N12 15	10 05	Regulus
Fomalhaut	1.3	16 22	S29 55	218 48	S 8 25	9 25	Alphard
Hamal	2.2	329 01	N23 12	(221 52)	S69 29	9 13	Miaplacidus
Kaus Aust.	2.0	84 54	S34 25	223 32	S43 12	9 06	Al Suhail
Kochab	2.2	(137 17)	N74 24	234 40	S59 20	8 21	ε Argus
Marfak	1.9	309 56	N49 39	244 33	N28 10	7 42	Pollux
Markab	2.6	14 31	N14 54	245 55	N 5 22	7 36	Procyon
Miaplacidus	1.8	(221 52)	S69 29	255 54	S28 54	6 56	Adhara
Mizar	2.4	159 35	N55 14	259 21	S16 38	6 43	Sirius
Nunki	2.1	77 04	S26 22	264 20	S52 40	6 23	Canopus
Peacock	2.1	54 42	S56 55	271 59	N 7 24	5 52	Betelgeux
Polaris	2.1	(333 51)	N88 59	276 40	S 1 14	5 33	Alnilam
Pollux	1.2	244 33	N28 10	279 29	N 6 18	5 22	Bellatrix
Procyon	0.5	245 55	N 5 22	281 53	N45 56	5 12	Capella
Rasalague	2.1	96 55	N12 36	282 03	S 8 16	5 12	Rigel
Regulus	1.3	208 40	N12 15	291 50	N16 24	4 33	Aldebaran
Rigel	0.3	282 03	S 8 16	309 56	N49 39	3 20	Marfak
Rigil Kent.	0.3	141 04	S60 36	315 59	S40 32	2 56	Acamar
Ruchbah	2.8	339 29	N59 56	329 01	N23 12	2 04	Hamal
Sabik	2.6	103 13	S15 39	(333 51)	N88 59	1 45	Polaris
Shaula	1.7	97 33	S37 04	336 06	S57 31	1 36	Achernar
Sirius	− 1.6	259 21	S16 38	339 29	N59 56	1 22	Ruchbah
Spica	1.2	159 27	S10 52	349 49	S18 18	0 41	Deneb Kait.
α Tri. Aust.	1.9	(109 20)	S68 56	358 28	N58 50	0 06	Caph
Vega	0.1	81 14	N38 44	358 38	N28 47	0 05	Alpheratz

SHA = 360° − RA   GHA* = GHA ♈ + SHA*   May–Aug., 1943

Note: Certain "index numbers" and cross reference symbols have been omitted from the first column of this table.

POLARIS, 1943

APPARENT PLACE, TIME OF UPPER CULMINATION, AND TIME INTERVAL BETWEEN UPPER CULMINATION AND ELONGATION EAST OR WEST

[From the *Nautical Almanac*]

The local civil time of culmination on any meridian for a given date is found by taking from the following table the *Civil Time* of the nearest Greenwich culmination and reducing it to the given date by means of the *Var. per Day*, and to the longitude of the given meridian by means of the *Var. per Hour.*

The time interval between upper and lower culmination is 12^h diminished by one-half the numerical value of the *Var. per Day.*

The last column below applies to all meridians.

Civil Time	Upper Culmination, Meridian of Greenwich					Latitude	Mean Time Interval, Elongation *minus* Upper Culm.
	Apparent Right Ascension	Apparent Declination	Civil Time	Var. per Day	Var. per Hour		
	h m 1 43	° ′ +88 59			W. E.		W. E.
	s	″	*h m s*	*m s*	*s*	°	*h m*
Jan. 0.8	128	50.9	19 6 51	− 3 57.0	− 9.88 +	10	+ 5 58.3 −
10.8	116	52.1	18 27 20	3 57.1	9.88	12	5 58.2
20.7	104	52.7	17 47 48	3 57.2	9.88	14	5 58.0
30.7	91	52.7	17 8 17	3 57.1	9.88	16	5 57.9
Feb. 9.7	79	51.9	16 28 45	3 57.1	9.88	18	5 57.7
19.7	68	50.6	15 49 15	− 3 57.0	− 9.87 +	20	+ 5 57.6 −
Mar. 1.6	57	48.7	15 9 46	3 56.8	9.87	22	5 57.4
11.6	49	46.4	14 30 18	3 56.7	9.86	24	5 57.2
21.6	42	43.6	13 50 53	3 56.4	9.85	26	5 57.1
31.5	38	40.6	13 11 29	3 56.2	9.84	28	5 56.9
Apr. 10.5	36	37.5	12 32 8	− 3 56.0	− 9.83 +	30	+ 5 56.7 −
20.5	36	34.4	11 52 49	3 55.8	9.82	32	5 56.5
30.5	39	31.4	11 13 33	3 55.5	9.81	34	5 56.3
May 10.4	44	28.6	10 34 18	3 55.3	9.81	36	5 56.1
20.4	50	26.2	9 55 6	3 55.2	9.80	38	5 55.9
30.4	59	24.1	9 15 55	− 3 55.0	− 9.79 +	40	+ 5 55.7 −
June 9.4	68	22.4	8 36 46	3 54.9	9.79	42	5 55.4
19.3	79	21.3	7 57 38	3 54.8	9.78	44	5 55.1
29.3	91	20.7	7 18 30	3 54.7	9.78	46	5 54.9
July 9.3	103	20.6	6 39 23	3 54.7	9.78	48	5 54.6
19.2	116	21.0	6 0 17	− 3 54.7	− 9.78 +	50	+ 5 54.2 −
29.2	128	22.0	5 21 10	3 54.7	9.78	52	5 53.9
Aug. 8.2	140	23.5	4 42 3	3 54.8	9.78	54	5 53.5
18.2	151	25.5	4 2 55	3 54.8	9.78	56	5 53.1
28.1	162	27.9	3 23 46	3 54.9	9.79	58	5 52.6
Sept. 7.1	171	30.7	2 44 36	− 3 55.0	− 9.79 +	60	+ 5 52.1 −
17.1	179	33.8	2 5 25	3 55.2	9.80	62	5 51.5
27.1	186	37.2	1 26 13	3 55.3	9.81	64	5 50.8
Oct. 7.0	191	40.8	0 46 59	3 55.5	9.81	66	5 50.0
17.0	194	44.5	0 7 43	3 55.7	9.82	68	5 49.1
26.9	195	48.3	23 28 25	− 3 55.9	− 9.83 +	70	+ 5 48.0 −
Nov. 5.9	194	52.1	22 49 5	3 56.1	9.84		
15.9	192	55.7	22 9 43	3 56.3	9.85		
25.9	187	59.2	21 30 20	3 56.5	9.85		
Dec. 5.9	180	62.3	20 50 54	3 56.7	9.86		
15.8	172	65.0	20 11 26	− 3 56.8	− 9.87 +		
25.8	162	67.2	19 31 57	3 57.0	9.87		
35.8	150	68.8	18 52 27	− 3 57.1	− 9.88 +		

NAVIGATIONAL STAR CHART

The purpose of the Star Chart facing this page is to assist the navigator in identifying stars for navigation. The stars of each constellation are connected by dotted lines; the bright stars are identified by their Greek letters and the principal stars by their common names. The Sidereal Hour Angle and declination of each star can be determined from the map by means of the network of vertical and horizontal lines drawn upon it for this purpose. The SHA's are measured (0° to 360°) from the vertical line passing through the Vernal Equinox at 0°. The declinations are measured North and South (N 90° to S 90°) from the celestial equator which is represented by a heavy horizontal line through the center.

The user of the star chart should be forewarned that the rectangular shape of the chart distorts the relative positions of the stars in the polar regions. A globe would give a better representation, and an observer from the inside would see the constellations as they appear in the sky. Then the SHA lines would converge at the North and South poles and the equator and ecliptic would be in the form of circles.

[Any strangeness due to the eastern edge of the chart's being on the left of north will be dissipated by holding the chart overhead with the northern edge toward Polaris.]

An observer's local meridian is easily located on the chart since it coincides with the vertical line whose GHA is equal to his longitude. GHA is not given directly on the chart but may be readily obtained from the SHA which is given. For any given instant GHA may be obtained from SHA by adding the GHA ♈ from the daily sheet:

$$\text{GHA} = \text{SHA} + \text{GHA}\ \Upsilon$$

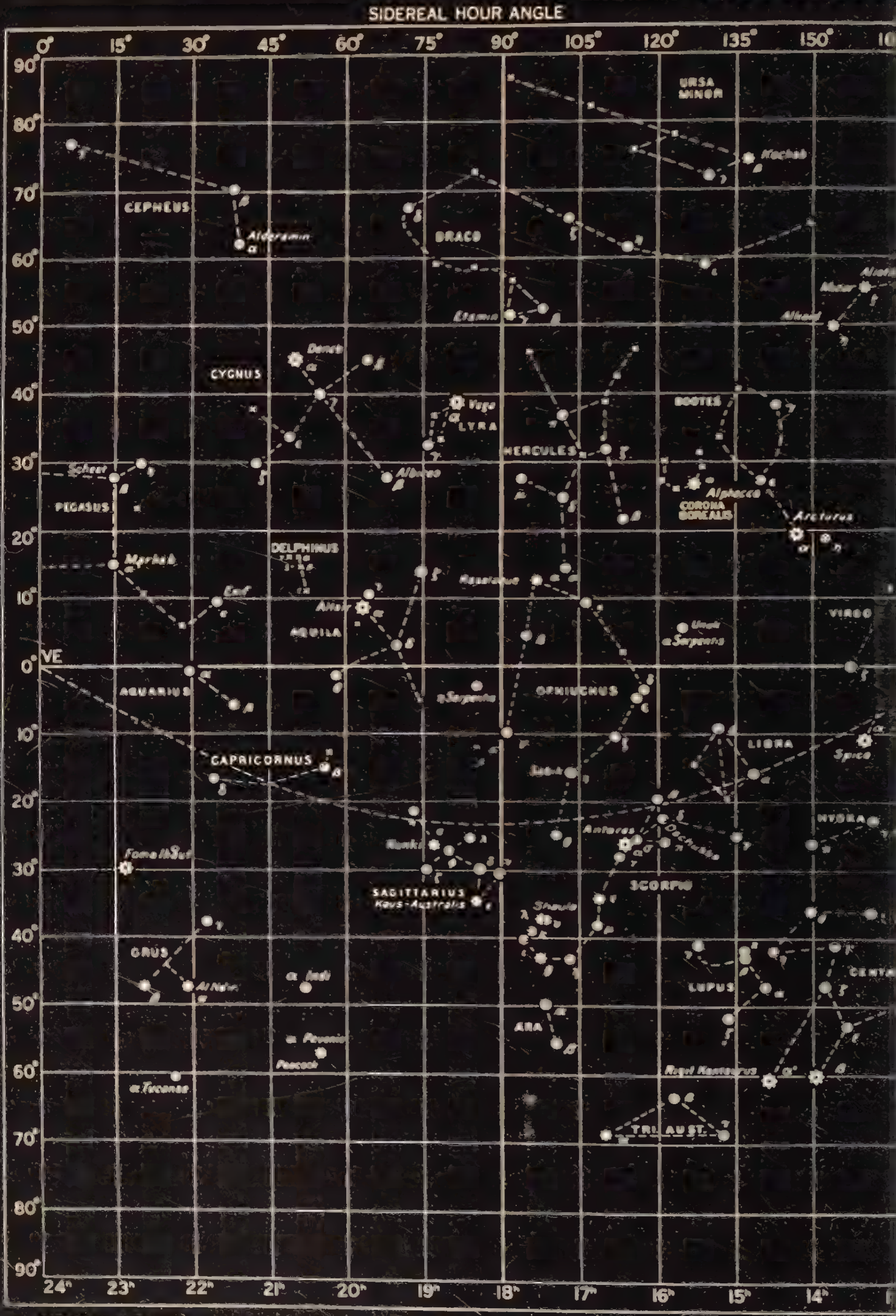
$$\text{Conversely}\qquad \text{SHA} = \text{GHA} - \text{GHA}\ \Upsilon$$

[Example: Locate on the chart the local meridian of an observer in longitude 63° E. on Aug. 1, 1943, at $18^h\ 10^m$ GCT. Since his longitude is 63° E., the GHA of his meridian will be − 63° or 297°. From the daily sheet for Aug. 1 at $18^h\ 10^m$ the GHA ♈ = 222°. The SHA of his meridian will therefore be 297° − 222° = 75°.]

The identification of a star directly overhead; i.e., in the zenith, is easily made since the point overhead is on the local meridian and also has a Dec. equal to the observer's latitude.

[Example: Assume that the observer in the above example is in latitude 40° N and that a star in the zenith is to be identified. The SHA of the star is exactly equal to the SHA of the local meridian and was

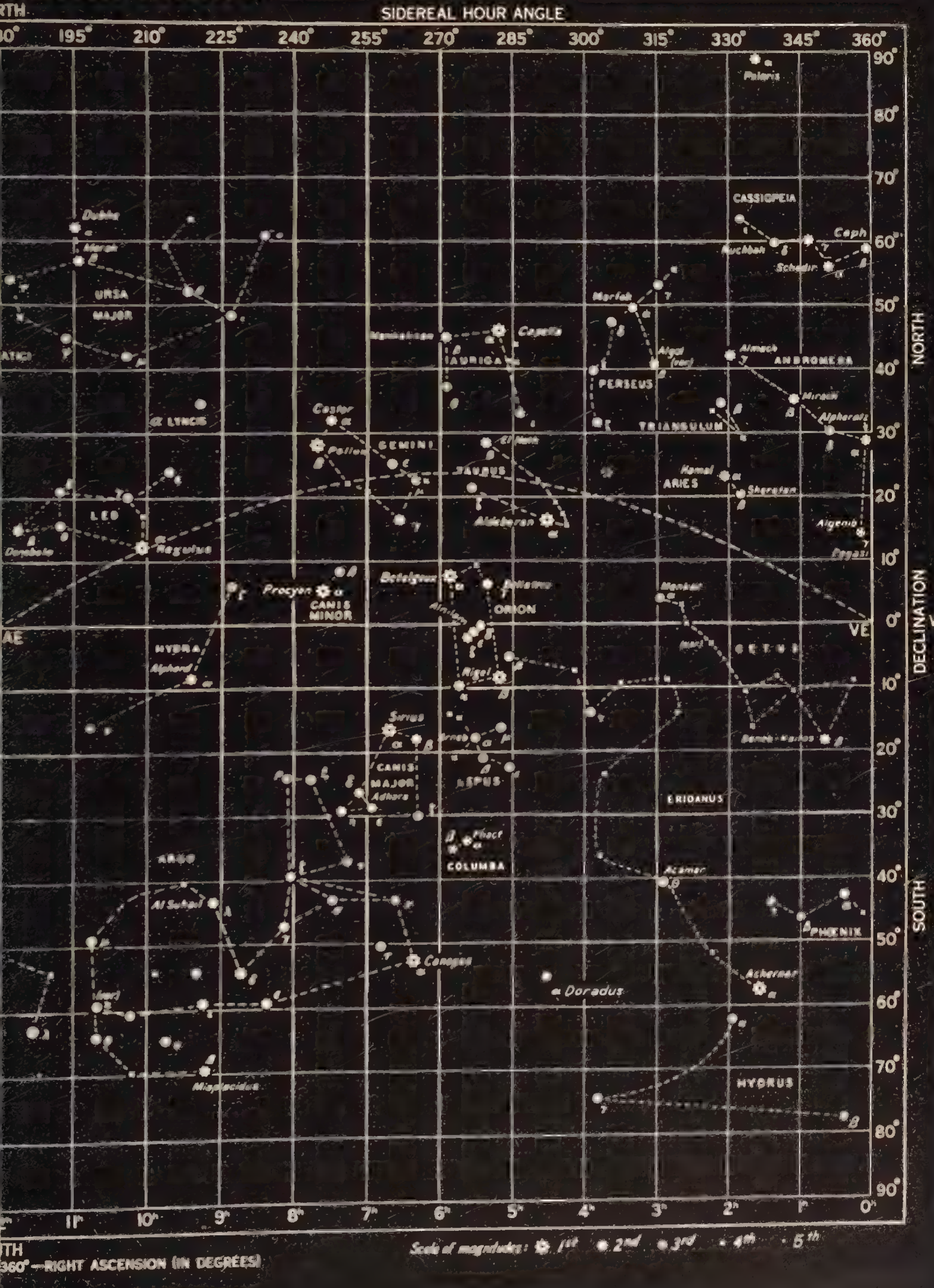
NAVIGATI



VE - Vernal Equinox
AE - Autumnal Equinox

SIDEREAL HOU

L STAR CHART


RTH
SIDEREAL HOUR ANGLE
195° 210° 225° 240° 255° 270° 285° 300° 315° 330° 345° 360°
90° 80° 70° 60° 50° 40° 30° 20° 10° 0° 10° 20° 30° 40° 50° 60° 70° 80° 90°
NORTH
DECLINATION
W
SOUTH
Polaris
CASSIOPEIA
Schedir
Dubhe
Merak
URSA
MAJOR
α LYNCIS
PERSEUS
ANDROMEDA
Alpheratz
TRIANGULUM
AURIGA
Capella
Castor
Pollux
GEMINI
TAURUS
Aldebaran
ARIES
Hamal
Sheratan
LEO
Regulus
Denebola
Procyon
CANIS
MINOR
Betelgeux
Bellatrix
ORION
Menkar
VE
AE
HYDRA
Alphard
CETUS
Rigel
Sirius
CANIS
MAJOR
Adhara
LEPUS
ERIDANUS
COLUMBA
Phact
ARGO
Al Suhail
Canopus
Acamar
PHENIX
Achernar
α Doradus
Miaplacidus
HYDRUS
11h 10h 9h 8h 7h 6h 5h 4h 3h 2h 1h 0h
Scale of magnitudes: 1st 2nd 3rd 4th 5th
TH
360°—RIGHT ASCENSION (IN DEGREES)

found to be 75°. The star's Dec. is N 40° since the observer's latitude is equal to the Dec. of a point in the zenith.

Examination of the chart in the region of SHA = 75° and Dec. N 40° shows the brightest star in the region to be the first magnitude star Vega. To verify, this region of the chart may be compared in detail with the sky. One finds the conspicuous star Altair 30° south and a little east of the zenith.]

A star to the North or South of the zenith is easily identified because its angular distance from the zenith is equal to the difference between its declination and the observer's latitude. [Thus, in the above example, Nunki (Dec. S 26°) would appear about 66° south of the zenith or at an altitude of 24°.]

* * * * * *

The Ecliptic, which if shown in the diagram on the daily sheet would be a straight line, is represented on the chart by a curved dotted line. The four bright stars of the diagram are easily found on the chart as they lie along the Ecliptic. The Sun, Moon, and planets may be plotted on the chart by means of their SHA and Dec.

[EXAMPLE: The daily sheet for Aug. 1, 0^h GCT gives GHA Mars = 265°, GHA ♈ = 309°, and Dec. Mars = N 15°. This gives SHA Mars = −44° = 316°. Plotting SHA and Dec. places Mars on the Ecliptic about 20° west of Aldebaran, which agrees with the daily diagram.]

Answers to Odd Problems

(Tables used have been *Useful Tables*: H.O. No. 9, Part II; United States Government Printing Office, Washington, D.C. Interpolation has arbitrarily been performed always from the smaller angle, and supplementary angles have arbitrarily been found always directly from the tables. For angles between 0° and 40′ the relations

$$\tan \theta = \sin \theta = (\text{number of minutes in } \theta)(\sin 1')$$

have been used. Other tables and other conventions may lead to answers differing from those below by a few seconds. The term "Odd Problems" refers to (1) all odd-numbered problems having but one part and (2) all odd parts of odd- and even-numbered problems when all the parts are of the same type of problem.)

Section 10, pages 48-49. (1) $\cos^{-1}(1/\sqrt{3}) = 54° 44'$. (3) $\cos^{-1}(-\sqrt{3}/4) = 115° 40'$, $\cos^{-1}(-5/8) = 128° 41'$, $\cos^{-1}(2\sqrt{3}/5) = 46° 09'$.

Section 13, pages 53-54. (1) 150°. (3) Leg: 45°; Angles: 60°, $\cos^{-1}(\sqrt{6}/4)$. (9) 45°, 135°.

Section 15, pages 58-59. 3 (a) $a = 28° 00'$, $b = 168° 50'$, $B = 157° 12'$. (c) $a = 121° 10'$, $B = 41° 20'$, $c = 114° 25'$. (e) $a = 114° 50'$, $A = 107° 56'$, $B = 132° 50'$. 4 (a) $A = 82° 23' 34''$, $B = 20° 08' 18''$, $c = 68° 38' 28''$. (c) $a = 51° 33' 38''$, $b = 30° 51' 08''$, $c = 57° 44' 33''$. (e) $a = 00° 14' 18''$ $A = 03° 15' 23''$, $b = 175° 48' 34''$.

Section 18, pages 67-68. (1) Hypotenuse: 45°; Leg: $\sin^{-1}(\sqrt{2}/4)$ in II; Angle: 150°; Hypotenuse: 135°; Leg: $\sin^{-1}(\sqrt{2}/4)$ in I; Angle: 30°.

3	Hypotenuse	Leg	Angle
(a)	64° 15′,	61° 15′,	77° 00′;
	115° 45′,	118° 45′,	103° 00′.
(c)	09° 50′,	09° 05′,	67° 45′;
	170° 10′,	170° 55′,	112° 15′.
(e)	90° 00′,	90° 00′,	90° 00′.

4	Hypotenuse	Leg	Angle
(a)	57° 12′ 15″,	42° 57′ 18″,	54° 09′ 46″;
	122° 47′ 44″,	137° 02′ 40″,	125° 50′ 13″.
(c)	61° 55′ 26″,	173° 14′ 10″,	172° 19′ 44″;
	118° 04′ 35″,	06° 45′ 47″,	07° 40′ 14″.
(e)	23° 32′ 02″,	05° 04′ 26″,	12° 47′ 41″;
	156° 27′ 58″,	174° 55′ 39″,	167° 12′ 18″.

5 (a) $A = c = 90°$, $b = 132° 14' 47''$. (c) No solution. (e) No solution. (g) $a = c = A = 90°$. (i) No solution. (k) $b = 55° 20'$, $A = 132° 55'$, $B = 65° 20'$.

Section 21, pages 70-71. (1) Vertex angle: 90°; Base angles: $\cos^{-1}(1/\sqrt{3}) = 54° 44'$. (3) $\cos^{-1}(\sqrt{2}/3) = 61° 52'$, $\cos^{-1}(1/6) = 80° 24'$, $\cos^{-1}(-\sqrt{2}/4)$

$= 110^\circ 42'$. (5) $\cos^{-1}(1/3) = 70^\circ 32'$. (7) Sides: $\cot^{-1}(\sqrt{3}/2) = 49^\circ 09'$, $\cot^{-1}(-1/\sqrt{6}) = 112^\circ 12'$, Angle: $\cos^{-1}(\sqrt{2}/4) = 69^\circ 18'$. (9) 0.347. 10 (a) $A = 33^\circ 00'$, $B = 45^\circ 18'$, $C = 147^\circ 00'$. (c) $A = B = C = 137^\circ 05'$. (e) $A = 127^\circ 00'$, $b = c = 67^\circ 54'$. (g) $B_1 = 25^\circ 20'$, $c_1 = 39^\circ 50'$, $C_1 = 15^\circ 55'$; $B_2 = 154^\circ 40'$, $c_2 = 140^\circ 10'$, $C_2 = 164^\circ 05'$. 11 (a) $A = 42^\circ 54' 16''$, $B = 45^\circ 55' 37''$, $C = 161^\circ 44' 00''$. (c) $A = 150^\circ 53' 04''$, $b = c = 151^\circ 13' 02''$. (e) $a = b = c = 31^\circ 41' 24''$.

Section 23, pages 72-73. 4 (a) $a = A = c = 90^\circ$. (c) No solution. (e) No solution. (g) $A = 129^\circ 55'$, $b = 44^\circ 20'$, $c = 123^\circ 15'$. (i) $a = c = 90^\circ$, $B = 16^\circ 22'$. 5 (a) $b = B = c = 90^\circ$. (c) $b_1 = 05^\circ 19' 58''$, $B_1 = 25^\circ 24' 00''$, $c_1 = 167^\circ 29' 16''$; $b_2 = 174^\circ 40' 04''$, $B_2 = 154^\circ 36' 00''$, $c_2 = 12^\circ 30' 44''$. (e) No solution. (g) $A = 137^\circ 13' 22''$, $B = 96^\circ 53' 10''$, $c = 82^\circ 30' 01''$. (i) $a = 124^\circ 30' 21''$, $b = 76^\circ 29' 33''$, $B = 78^\circ 48' 12''$.

Section 26, page 80. 1 (a) $A = 79^\circ 30'$, $b = 50^\circ 50'$, $C = 44^\circ 20'$. (c) $A = 41^\circ 10'$, $B = 131^\circ 51'$, $c = 34^\circ 30'$. (e) $A = 90^\circ 00'$, $B = 110^\circ 25'$, $c = 120^\circ 20'$. 2 (a) $A = 38^\circ 26' 58''$, $b = 41^\circ 44' 27''$, $C = 88^\circ 27' 02''$. (c) $A = 59^\circ 32' 31''$, $b = 60^\circ 10' 42''$, $c = 96^\circ 57' 25''$. (e) $A = 140^\circ 42' 43''$, $b = 41^\circ 21' 48''$, $C = 125^\circ 41' 09''$. (g) $a = 23^\circ 00' 32''$, $B = 101^\circ 36' 06''$, $c = 146^\circ 57' 32''$. (3) $104^\circ 14' 46''$. (5) $63^\circ 26' 36''$. (7) $90^\circ 00' 00''$.

Section 28, page 88. 1 (a) $a_1 = 109^\circ 20'$, $B_1 = 57^\circ 40'$, $A_1 = 123^\circ 02'$; $a_2 = 14^\circ 00'$, $B_2 = 122^\circ 20'$, $A_2 = 12^\circ 22'$. (c) $a = 141^\circ 35'$, $A = 135^\circ 05'$, $B = 90^\circ 00'$. (e) No solution. 2 (a) $a = 162^\circ 54' 27''$, $c = 07^\circ 11' 19''$, $C = 24^\circ 25' 07''$. (c) No solution. (e) $a_1 = 17^\circ 23' 27''$, $A_1 = 128^\circ 57' 32''$, $B_1 = 149^\circ 59' 53''$. $a_2 = 03^\circ 14' 15''$, $A_2 = 08^\circ 27' 44''$, $B_2 = 30^\circ 00' 08''$. (g) $a_1 = 150^\circ 40' 54''$, $c_1 = 41^\circ 08' 37''$, $C_1 = 154^\circ 18' 24''$. $a_2 = 29^\circ 19' 08''$, $c_2 = 163^\circ 01' 29''$, $C_2 = 168^\circ 55' 24''$. (3) $15^\circ 57' 50''$. (5) 0.052.

Section 30, page 93. 1 (a) $A = 61^\circ 30'$, $B = 96^\circ 25'$, $C = 119^\circ 53'$. (c) $A = 51^\circ 10'$, $B = 72^\circ 09'$, $C = 91^\circ 05'$. 2 (a) $A = 66^\circ 44' 44''$, $B = 98^\circ 57' 27''$, $C = 118^\circ 44' 09''$. (c) $a = 24^\circ 15' 32''$, $b = 129^\circ 19' 20''$, $c = 132^\circ 16' 37''$. (e) $A = 44^\circ 07' 24''$, $B = 97^\circ 59' 56''$, $C = 118^\circ 58' 05''$. (3) $73^\circ 26'$. (5) $57^\circ 44' 48''$, or $122^\circ 15' 12''$.

Section 31, pages 93-94. 1 (a) $a = 104^\circ 33'$, $A = 75^\circ 27'$, $B = 152^\circ 00'$. (c) $A = 50^\circ 15'$, $B = 73^\circ 10'$, $C = 88^\circ 20'$. (e) $a = 31^\circ 45'$, $b = 131^\circ 45'$, $c = 154^\circ 00'$. (g) $a = 74^\circ 37'$, $b = 128^\circ 46'$, $c = 74^\circ 37'$. (i) $A = 139^\circ 05'$, $B = 139^\circ 05'$, $C = 139^\circ 05'$. 2 (a) $A = 23^\circ 02' 44''$, $B = 76^\circ 00' 40''$, $C = 100^\circ 07' 10''$. (c) $B_1 = 157^\circ 28' 08''$, $c_1 = 176^\circ 01' 20''$, $C_1 = 173^\circ 56' 52''$. $B_2 = 22^\circ 31' 54''$, $c_2 = 155^\circ 55' 20''$, $C_2 = 38^\circ 16' 48''$. (e) $A = 161^\circ 22' 44''$, $B = 135^\circ 04' 48''$, $c = 84^\circ 23' 19''$. (g) $A = 172^\circ 48' 30''$, $B = 110^\circ 37' 44''$, $c = 92^\circ 43' 22''$. (i) No solution. (k) $a = 48^\circ 07' 26''$, $b = 136^\circ 56' 05''$, $c = 152^\circ 11' 39''$. (m) $a = 25^\circ 28' 50''$, $A = 88^\circ 29' 02''$, $B = 47^\circ 15' 52''$. (o) No solution. (q) $A = B = C = 135^\circ 40' 10''$. (s) $a = 58^\circ 19' 31''$, $c = 97^\circ 54' 15''$, $C = 82^\circ 05' 54''$. (u) $b = 56^\circ 50' 52''$, $B = 63^\circ 25' 04''$, $C = 90^\circ 00' 00''$. (3) $32^\circ 28' 15''$. (5) $37^\circ 37' 38''$ and $73^\circ 58' 38''$. (7) $40^\circ 46' 28''$.

Section 36, pages 101-102. (3) 415 statute miles. (5) $03^\circ 38' 19''$, $01^\circ 26' 52''$. (7) 0°, 180°, 20.7 hours, 525 gallons.

Section 38, pages 106-108. (1) $66^\circ 04' 48''$, $47^\circ 01' 52''$ W. (7) $62^\circ 38' 22''$. (9) lat. $37^\circ 45'$ N., long. $74^\circ 07'$ E. 10 (a) 32.6 nautical miles; $71^\circ 53' 24''$; $108^\circ 06' 36''$; $48^\circ 20' 53''$ N. (c) 19.6 nautical miles; $287^\circ 04' 24''$; $252^\circ 55' 36''$;

57° 27′ 06″ N. 11 (a) Right; 11.746 nautical miles. (c) Left; 14.597 nautical miles. (e) Left; 184.78 nautical miles. (g) Left; 28.393 nautical miles. (13) 49° 21′ 24″ N.

Section 49, pages 135, 138-139. 2 (a) $23^h 20^m 12^s$. (c) $3^h 04^m 28^s$. 4 (a) (1) Alioth, Caph; (2) Pollux, Fomalhaut; (3) Canopus. (c) (1) Polaris; (2) Acrux, Dubhe, α Tri. Aust., θ Centauri; (3) none. 5 (a) $t = 2^h\pm$; $A_z = 120°\pm$ west of north. (c) $t = 23^h\pm$; $A_z = 45°\pm$ east of north. (e) $t = 6^h\pm$; $A_z = 135°\pm$ west of north. 6 (a) 270°. (c) 135°±. (7) $|\text{lat.} - \text{d}| < 90°$, where lat. and d are signed (+ if N., − if S.) 8 (a) Polaris, Kochab. 9 (a) South of latitude 30° 04′ S. (e) North of latitude 33° 05′ N. (c) North of latitude 30° 40′ N. (g) South of latitude 44° 04′ S. 10 (a) $4^h 05^m$, $16^h 05^m$. (c) $23^h 52^m$, $11^h 52^m$.

Section 51, pages 142-143. (1) d of same sign as lat., and $|\text{d}| > 90° - |\text{lat.}|$. (3) Decreases as latitude increases. 4 (a) 60° N., (c) 44° 10′ N. (5) The higher the latitude the greater the number of circumpolar stars. $|\text{d}| > 90° - |\text{lat.}|$, and d and lat. of same sign. 7 (a) 01° 03′ 07″ counterclockwise, $18^h 01^m 10^s$. (c) 01° 32′ 51″ clockwise, $5^h 55^m 19^s$. 8 (a) 59° 09′ 20″ S. (c) 50° 59′ 30″ S. (e) 58° 34′ 40″ N.

Section 53, pages 150-152. (3) $20^h 34^m 56^s$, $8^h 36^m 52^s$, $1^m 56^s$. 4 (a) Early April. (c) Late November or early December. 5 (a) 9:10 A.M., $9^h 14^m 40^s$. (c) 7:20 A.M., $7^h 39^m 00^s$. 7 (a) $16^h 28^m 08^s$. (c) $1^h 09^m 52^s$. 9 (a) lat. 35° 44′ 30″ S., long. 19° 10′ E. (c) lat. 09° 56′ S., long. 63° 33′ E. (e) lat. 08° 35′ N., long. 88° 29′ W. 10 (a) $15^m 05^s - 08^s$ fast. 11 (a) Four times. Minus for 3 months after Christmas; Plus for next 3 months; Minus for following 3 months; Plus for 3 months before Christmas. (b) Twice. Minus for 6 months after Christmas; Plus for 6 months before Christmas. (c) Four times. (d) During the 6 months straddling Christmas the two causes of equation of time are working together; during the other 6 months the two causes oppose one another.

Section 56, pages 162-167. 1 (a) $t = 21^h 00^m 17^s$, $A_z = 128° 15′ 26″$ east of north. (c) lat. = 14° 55′ 32″ N., $t = 1^h 09^m 09^s$. (e) $h = 22° 54′ 02″$, $A_z = 132° 58′ 06″$ west of south. (g) lat. = 39° 55′ 16″ S., $A_z = 37° 02′ 14″$ west of south, or lat. = 80° 39′ 16″ S., $A_z = 142° 57′ 48″$ west of south. 2 (a) 62° 05′ 14″ west of south, 13.9 nautical miles. (c) 69° 26′ 42″ west of north, 15.4 nautical miles. (e) 53° 19′ 41″ west of north, 11.9 nautical miles. (g) 117° 12′ 00″ east of north, 2.7 nautical miles. 3 (a) 2:56:22 P.M. (c) 1:06:38 P.M. 4 (a) $5^h 41^m 17^s$, 82° 59′ 00″ west of south. (c) $2^h 33^m 04^s$, 35° 30′ 13″ west of south. (e) $4^h 55^m 44^s.5$, 63° 06′ 00″ west of north. (g) $5^h 59^m 59^s.9$, 89° 50′ 00″ west of north. (i) $9^h 53^m 22^s$, 29° 54′ 00″ west of north. 5 (a) 48° 55′ 16″ east of north. (c) 43° 05′ 50″ east of north. (e) 75° 42′ 38″ east of south. (g) 89° 56′ 58″ east of north. (i) No shadow, as sun has set. 6 (a) 41° 19′ 08″ W. (c) 136° 50′ 57″ W. (9) 8:45:20 P.M., western, 01° 16′ 39″ clockwise. 10 (a) 4:22:15 A.M., western, 01° 04′ 59″ clockwise. (c) 3:55:17 A.M., western, 01° 54′ 15″ clockwise. 11 (a) 55° 56′ 38″ N. (c) 57° 21′ 24″ N. or 24° 48′ 52″ N. (in which case a bubble octant must have been used). (13) 36° 32′ 22″ N. or S. (15) Boston: mean, apparent, legal; New York: mean, legal, apparent; Charleston: legal, mean, apparent. (17) $8^h 59^m 54^s - 57^s$. (19) 77° 18′ 07″ southward or counterclockwise. 20 (a) $3^h 09^m 13^s$, $3^h 15^m 29^s$, 3:43:21 A.M. (c) $4^h 48^m 14^s$, $4^h 54^m 30^s$, 5:04:50 A.M. 21 (a) lat. 45° 05′ N., long. 93° 30′ W.; Minneapolis, Minn. (c) lat. 34° 25′ S., long. 58° 27′ W.; Buenos Aires. 23 (a) 7:45:02 − 05 P.M. 25 (a) The shadow of such a stick measures the sun's azimuth; to form a sundial it should measure the sun's hour angle. Yes, either terrestrial pole. Such a shadow would be of constant direction (west) all morning and of constant direction (east) all afternoon. (b) - - - -. Local apparent time. The ray for n-hours before or after noon must make an angle of n 15° with the north line in the face, to the west for morning, to the east for afternoon. (c) $\tan\theta_n = (\sin \text{lat.})(\tan n\,15°)$. Usable only in latitudes equal to the angle at which the style is inclined to the face. (d) $\tan\theta_n = (\cos \text{lat.})(\tan n\,15°)$. Style will point to the depressed pole. Will not be usable the year around for points in the torrid zone. (e) 49° 57′ 02″.

Section 13 (Appendix II), pages 203-204. (1) $A = 21° 23' 56''$, $B = 31° 59' 56''$, $C = 163° 41' 14''$. (3) 6445.2 nautical miles, 240° 17′ 10″. (5) $AB = 2704.9$ nautical miles, $PB = 1687.4$ nautical miles. (7) 120° 04′ 52½″. (9) $B = 44° 12' 09''$, $A = 144° 01' 10''$. (11) 100° 44′ 05″. (13) $18^h\,55^m\,12^s$, 28° 19′ 02″, 972 nautical miles. (15) 222° 40′ 25″, 6111.8 nautical miles.

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